DOCTORAL THESIS

A General Estimation Framework for Nonlinear Singularly Perturbed Systems

Author
Luis Angel Cuevas Ramirez
ORCID: 0000-0002-1763-4099

Supervisors
Prof. Dragan Nešić
Prof. Christopher Manzie

Submitted in partial fulfilment of the requirements of the degree of
Doctor of Philosophy

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
THE UNIVERSITY OF MELBOURNE

July 2019

Abstract

ESTIMATION of unmeasured variables is a central objective in a broad range of applications. However, the estimation process turns into a challenging task when the underlying model is nonlinear and even more so when additionally it exhibits multiple time-scales. The current results on estimation for systems with two time-scales apply to linear systems, and limited classes of nonlinear plants and specific observers. Therefore, a new and robust estimation framework for nonlinear systems with variables evolving in different time-scales is needed. This work focuses on developing a rigorous theoretical body for the state estimation of a general class of nonlinear singularly perturbed systems by assuming that the input and output are measured.

In the first part of the thesis, we consider the estimation of the slow state of globally Lipschitz nonlinear singularly perturbed systems by using a full-order observer synthesised for the reduced order (slow) model. We deal with the case when the measured output is disturbed by bounded measurement noise. We prove a global exponential input-to-state (ISS) practical stability property for the estimation error with ISS gain from the measurement noise. Moreover, we show that our assumptions are such that they also imply practical $\mathcal{L}_2$ stability of the error dynamics. Our findings apply to a general class of nonlinear globally Lipschitz singularly perturbed systems, and to a number of full-order observers. In order to prove the robustness results to singular perturbations and to measurement noise of the observer, we first show that the plant has bounded solutions under an appropriate set of assumptions on the corresponding boundary layer and reduced order models. We demonstrate the applicability of our findings by showing that the stated assumptions hold for at least four classes of plants and nonlinear observers. Moreover, we present simulation results for numerical examples.

In the second part of the thesis, we generalise current results in the literature and the results of the first part by considering broader classes of plants and estimators of general dimension to cover reduced-order, full-order and higher-order observers. Similarly

to the first part of the thesis, we first prove a boundedness result for the plant based on a set of assumptions imposed on the reduced order and boundary layer systems. We then exploit this boundedness property of the plant to show that the error dynamics of the observer designed for the reduced system are semi-globally input-to-state practically (ISpS) stable when the observer is implemented on the original plant. Furthermore, we conclude $\mathcal{L}_\infty \cap \mathcal{L}_2$ stability results when the measurement noise belongs to $\mathcal{L}_\infty \cap \mathcal{L}_2$. In the absence of measurement noise, we state results on semi-global practical asymptotical (SPA) stability for the error dynamics. We illustrate the generality of our main results through four classes of systems with corresponding observers.

In the third part of this thesis, we address the parameter and state estimation problem of nonlinear systems with unknown slowly time-varying parameters where the unknown parameter is assumed to belong to a compact set. We tackle this problem by using a multi-observer approach under the supervisory framework. This estimation technique requires a finite number of sample points taken from the compact set where the unknown parameter belongs to. Then, by using these samples as potential parameter estimates, a state observer is designed for each sample to construct a bank of observers to generate potential state estimates. The selection of the parameter and state estimates is performed under the supervisory framework by using a set of monitoring signals and a selection criterion. The monitoring signals characterise the quality of the output estimation errors so that the selection criterion chooses the estimate that gives the smallest difference between the measured and the estimated output.

In this thesis, we propose a novel dynamic sampling policy to generate the parameter samples. This new policy allows the application of the multi-observer technique on systems with slowly time-varying parameters so that our proposed approach is a non-trivial generalisation of the multi-observer technique for parameter and state estimation for systems with constant parameters. We prove that our proposed technique generates parameter and state estimates that are ultimately bounded where the ultimate bounds can be made arbitrarily small if the parameter is sufficiently slow, and if there is a sufficiently large number of observers. We have addressed the parameter and state estimation problem since the slow state of a singularly perturbed system can be regarded as a slowly time-varying parameter to the fast dynamics. Hence, the multi-observer technique for parameter and state estimation of nonlinear systems with unknown slowly time-varying parameters is natural to the singular perturbations framework.

Declaration

This is to certify that

1. the thesis comprises only my original work towards the PhD,
2. due acknowledgement has been made in the text to all other material used,
3. the thesis is less than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

Luis Angel Cuevas Ramirez, July 2019

Acknowledgements

First and foremost, I would like to express my sincere gratitude and appreciation to my principal supervisor Prof. Dragan Nešić. I am extremely grateful for his patience and for always having the right words to keep me motivated. I am thankful for his commitment to my development and his readiness. I admire his great enthusiasm for research, immense knowledge, gentleness and humility.

I would also like to thank my co-supervisor Chris Manzie for always showing me a new perspective for my research. His guidance, questions and suggestions helped me to enrich my work. I am sure that this research would have been impossible without the aid and support of Chris. I strongly believe that I could not have asked for better supervisors since Dragan and Chris motivated me to grow personally and professionally.

I am grateful to the academics and staff of the Department of Electrical and Electronic Engineering. Special thanks go to the Department Administrator Lyn Bunchanan since she has always been keen to help me in anything she can. I would also like to thank my advisory committee chair A/Prof. Peter Dower for his constructive comments and suggestions.

I am thankful to my friends and colleagues of the singular perturbations research team, Saeed Ahmadizadeh and Mohammad Deghat. I am extremely grateful to Saeed for helping me to put in track my PhD when I was loosing the wheel of my research. I will always remember Saeed's kindness in my moments of need and Mohammad's supportive words after each meeting to keep me positive about my research. Moreover, I am very pleased to give my sincere thanks my friends and colleagues in the Melbourne School of Engineering, including Carlos Murguia, Ricardo Garcia, Omin Monfred, Alejandro Maass, and Tianci Yang.

Finally, I would like to express my deep gratitude to my beloved family for their endless love, dedication and support. Special thanks to my mother Elda and my father David, I appreciate from my heart everything that you have done that has led me to

reach this milestone in my life. Special thanks go to my grandma Quina, my brothers Eduardo and David, and my family by choice Itzel, Carlos, Montse and Alma for their strong support in all aspects of my life during my PhD. Even though my grandpa Nico passed away a long time ago, my deepest gratitude goes to him because his last words to me brought me here.

Preface

The material presented in this thesis is an original research, unless otherwise stated, developed during my PhD candidature in the Department of Electrical and Electronic Engineering at The University of Melbourne. The work reported herein was conducted in collaboration with my supervisors Prof. Dragan Nešić and Prof. Chris Manzie. In addition, Dr. Romain Postoyan provided useful feedback for the problem formulation of Chapter 6. The results reported in this thesis have been accepted or submitted for publication in conferences and journals. For all these publications, I am the primary author and I was responsible primarily for the planning, execution, development and preparation of the work for publication. However, I benefited from my supervisors' feedback as they provided technical comments and guidance. The publications are listed in the following:

- L. Cuevas, D. Nešić, and C. Manzie. Convergence of full-order observers for the slow states of a singularly perturbed system (Part I: Theory). *Proceedings of 2018 Australian & New Zealand Control Conference (ANZCC)*, pp. 297 - 301, 2018.
- L. Cuevas, D. Nešić, and C. Manzie. Convergence of full-order observers for the slow states of a singularly perturbed system (Part II: Applications). *Proceedings of 2018 Australian & New Zealand Control Conference (ANZCC)*, pp. 352 - 357, 2018.
- L. Cuevas, D. Nešić, and C. Manzie. Global stability of the error dynamics of an observer designed for the slow states of a singularly perturbed system. *Proceedings of 2018 15th International Conference on Control, Automation, Robotics and Vision (ICARCV)*, pp. 698 - 703, 2018.
- L. Cuevas, D. Nešić, and C. Manzie. Robustness Analysis of Nonlinear Observers for the Slow Variables of Singularly Perturbed Systems. *Submitted to the International Journal of Robust and Nonlinear Control.*

Moreover, this thesis has generated results for two papers that are in preparation:

- L. Cuevas, D. Nešić, C. Manzie and R. Postoyan. Multi-observer approach for parameter and state estimation for systems with unknown slowly time-varying parameters. In preparation for journal submission.

- L. Cuevas, D. Nešić, C. Manzie and R. Postoyan. A new dynamic sampling policy for the multi-observer approach under the supervisory framework. In preparation for conference submission.

No portion of the work presented herein was conducted prior to my enrolment in the degree of Doctor of Philosophy at the University of Melbourne. Neither has any of it been submitted for any other qualification.

This research has been financially supported by the Melbourne International Research Scholarship scheme of The University of Melbourne, and the Australian Research Council (ARC) through the Discovery Project DP170104102. During my candidature, I was also supported by the Consejo Nacional de Ciencia y Tecnología (CONACYT) of the Mexican government. Furthermore, I obtained a travel grant from the Melbourne School of Engineering during my candidature.

To my grandpa Nico and to everyone who has motivated me, encouraged me, and believed in me during this amazing journey...

Contents

List of Figures

Nomenclature

$\mathbb{R}$	The set of real numbers.
$\mathbb{R}_{\geq 0}$	The set of non-negative real numbers.
$\mathbb{R}_{>0}$	The set of strictly positive real numbers.
$\mathbb{R}^{n \times m}$	The space of real matrices with dimensions $n \times m$.
$\mathbb{N}$	The set of non-negative integers.
$\mathbb{N}_{\geq 1}$	The set of positive integers.
$\lambda_{\min}\{A\}$	The minimum eigenvalue of a real, symmetric matrix A.
$\lambda_{\max}\{A\}$	The maximum eigenvalue of a real, symmetric matrix A.
$\star$	The symmetric block component of a symmetric matrix.
$\mathbb{I}$	The identity matrix.
$\lvert x \rvert$	The (Euclidean) norm of a vector $x \in \mathbb{R}^n$.
$\lvert x \rvert_\infty$	The ∞-norm of a vector $x = (x_1, x_2, \ldots, x_n)^\top \in \mathbb{R}^n$ which is defined as $\lvert x \rvert_\infty = \max_i \lvert x_i \rvert$, where $\lvert x_i \rvert$ denotes the absolute value.
$\mathcal{L}_\infty$	The set of functions $s : \mathbb{R} \to \mathbb{R}^n$, such that $\lVert s \rVert_\infty < \infty$, where $\lVert s \rVert_\infty := \text{ess sup}_t \lvert s(t) \rvert$.
$\lvert s[t_1, t_2] \rvert$	The supremum over the time interval $[t_1, t_2]$ of a function $s : \mathbb{R} \to \mathbb{R}^n$, i.e. $\lvert s[t_1, t_2] \rvert := \sup_{t \in [t_1, t_2]} \lvert s(t) \rvert$.
$\lvert s(t) \rvert_{\mathcal{L}_2}$	The $\mathcal{L}_2$-norm of $s : \mathbb{R} \to \mathbb{R}^n$ defined as $\lvert s(t) \rvert_{\mathcal{L}_2} := \sqrt{\int_0^\infty s(t)^\top s(t) dt}$.
$\mathbf{X}(x_c, \Delta)$	The hypercube centred at $x_c \in \mathbb{R}^n$ with distance to the edge $\Delta > 0$, i.e. $\mathbf{X}(x_c, \Delta) := \{x \in \mathbb{R}^n \mid \lvert x - x_c \rvert_\infty \leq \Delta\}$.
$d(x, \mathbf{X})$	The distance from $x \in \mathbb{R}^n$ to the elements of the set $\mathbf{X} \subset \mathbb{R}^n$ and it is defined as $d(x, \mathbf{X}) := \min_{\mathbf{x} \in \mathbf{X}} \lvert x - \mathbf{x} \rvert_\infty$.

$\mathcal{K}$	A continuous function $\alpha(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be a class-$\mathcal{K}$ function, if it is strictly increasing and $\alpha(0) = 0$.
$\mathcal{K}_\infty$	A continuous function $\alpha(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be a class-$\mathcal{K}_\infty$ function, if it is strictly increasing, $\alpha(0) = 0$, and additionally, $\alpha(s) \to \infty$ as $s \to \infty$.
$\mathcal{KL}$	A continuous function $\beta(\cdot,\cdot) : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be a class-$\mathcal{KL}$ function, if $\beta(\cdot, s)$ is a class-$\mathcal{K}$ function for each $s \geq 0$ and $\beta(r, \cdot)$ is non-increasing and $\beta(r, s) \to 0$ as $s \to \infty$ for each $r \geq 0$.
$(\cdot)^-$	The left-limit operator.
$\overset{(\text{No.})}{\leq}$	Indicates that the result on the right-hand side of the inequality follows from using equation\inequality (No.) on the left hand side of the inequality.

Fundamental preliminaries

The definitions of stability properties presented in here are essential for the full understanding and interpretation of the results of this thesis. I only give the definition of each property without having any discussions afterwards since they are widely known. I state fundamental stability definitions for systems with and without disturbances. Consider the non-autonomous system defined by

$$\dot{x} = f(t, x), \tag{1}$$

where $f : [0, \infty) \times D \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[0, \infty) \times D$, and $D \subset \mathbb{R}^n$ is a domain that contains the origin $x = 0$. We say that the origin is an equilibrium point for (1) at $t = 0$ if $f(t, 0) = 0$ for all $t \geq 0$ [70].

Definition 1. *[Definition 4.4, 70] The equilibrium point $x = 0$ of (1) is*

- *uniformly stable if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$, independent of t_0 such that*

$$|x(t_0)| < \delta \implies |x(t)| < \varepsilon, \ \forall\, t \geq t_0 \geq 0, \tag{2}$$

- *uniformly asymptotically stable if it is uniformly stable and there is a positive constant c, independent of t_0, such that for all $|x(t_0)| < c$, $x(t) \to 0$ as $t \to \infty$, uniformly in t_0; that is, for each $\eta > 0$, there is $T = T(\eta) > 0$ such that*

$$|x(t)| < \eta, \quad \forall\, t \geq t_0 + T(\eta), \quad \forall\, |x(t_0)| < c, \tag{3}$$

- *globally uniformly asymptotically stable if it is uniformly stable, $\delta(\varepsilon)$ can be chosen to satisfy $\lim_{\varepsilon \to \infty} \delta(\varepsilon) = \infty$, and, for each pair of positive numbers η and c, there is $T = T(\eta, c) > 0$ such that*

$$|x(t)| < \eta, \quad \forall\, t \geq t_0 + T(\eta, c), \quad \forall\, |x(t_0)| < c, \tag{4}$$

Definition 2. *[Lemma 4.5, 70] The equilibrium $x = 0$ of (1) is*

- *uniformly stable if and only if there exists a class-$\mathcal{K}$ function $\alpha(\cdot)$ and a positive constant c, independent of t_0, such that*

$$|x(t)| \leq \alpha(|x(t_0)|), \quad \forall\, t \geq t_0 \geq 0, \quad \forall\, |x(t_0)| < 0, \tag{5}$$

- *uniformly asymptotically stable if and only if there exists a class-$\mathcal{KL}$ function $\beta(\cdot, \cdot)$*

and a positive constant c*, independent of* t_0*, such that*

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0), \quad \forall\, t \geq t_0 \geq 0, \quad \forall\, |x(t_0)| < 0, \tag{6}$$

- *globally uniformly asymptotically stable if and only if inequality (6) is satisfied for any initial state* $x(t_0)$.

Definition 3. *[Definition 4.5, 70] The equilibrium point* $x = 0$ *of (1) is exponentially stable if there exists positive constants* c, k, λ *such that*

$$|x(t_0)| \leq k \exp[-\lambda(t - t_0)]|x(t_0)|, \quad \forall\, |x(t_0)| < c, \tag{7}$$

and globally exponentially stable if (7) is satisfied for any initial state $x(t_0)$.

Definition 4. *[Definition 4.6, 70] The solutions of (1) are*

- *uniformly bounded if there exists a positive constant* c*, independent of* $t_0 \geq 0$*, and for every* $a \in (0, c)$*, there is* $\beta = \beta(a) > 0$*, independent of* t_0*, such that*

$$|x(t_0)| < a \implies |x(t)| \leq \beta, \; \forall\, t \geq t_0, \tag{8}$$

- *globally uniformly bounded if (8) holds for arbitrarily large* a.
- *uniformly ultimately bounded with ultimate bound* b *if there exist positive constants* b *and* c*, independent of* $t_0 \geq 0$*, and for every* $a \in (0, c)$*, there is* $T = T(a, b) \geq 0$*, independent of* t_0*, such that*

$$|x(t_0)| < a \implies |x(t)| \leq b, \quad \forall\, t \geq t_0 + T, \tag{9}$$

- *globally uniformly ultimately bounded if (9) holds for arbitrarily large* a.

Now, consider the general class of nonlinear systems defined by

$$\dot{x} = f(x, u), \tag{10}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^r$ is the control input. Here, $f : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ is continuously differentiable and satisfies $f(0, 0) = 0$. For each $\xi \in \mathbb{R}^n$ and each $u \in \mathcal{L}_\infty$, we denote $x(t, \xi, u)$ the trajectory of the system (10) with initial state $x(0) = \xi$ and the input u. This is defined on some maximal interval $[0, T_{\xi,u})$, with $T_{\xi,u} \leq +\infty$ [116].

Definition 5. *[Definition 2.1, 116] The system (10) is (globally) input-to-state stable (ISS) if there exist a class-*$\mathcal{KL}$ *function* $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ *and a class-*$\mathcal{K}$ *function* $\gamma(\cdot)$ *such that, for each input* $u \in \mathcal{L}_\infty$ *and each* $\xi \in \mathbb{R}^n$ *it holds that*

$$|x(t, \xi, u)| \leq \beta(|\xi|, t) + \gamma(\|u\|_\infty), \tag{11}$$

for each $t \geq 0$.

Definition 6. *[Definition 2.1, 6] The system (10) is said to be* k*-th derivative input-to-state stable (D*^{k}*ISS) if there exist some class-*$\mathcal{KL}$ *function* $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ *and some class-*$\mathcal{K}$ *functions* $\gamma_0, \gamma_1 \ldots, \gamma_k$ *such that, for every input* $u \in \mathcal{L}_\infty$*, the following holds*

$$|x(t, \xi, u)| \leq \beta(|\xi|, t) + \gamma_0(\|u\|_\infty) + \gamma_0(\|\dot{u}\|_\infty) + \ldots + \gamma_k(\|u^{(k)}\|_\infty), \tag{12}$$

for all $t \geq 0$.

Definition 7. *The system (10) is said to be input-to-state practically stable (ISpS) if there exist a class-$\mathcal{KL}$ function* $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, *a class-$\mathcal{K}$ function* $\gamma(\cdot)$ *and a constant* $c \geq 0$,*such that*

$$|x(t, \xi, u)| \leq \beta(|\xi|, t) + \gamma(\|u\|_\infty) + c, \tag{13}$$

holds for each control u *and each* $\xi \in \mathbb{R}^n$ *and for all* $t \geq 0$ *[115].*

Definition 8. *[Definition 1, 94] The system* $\dot{x} = f(t, x, \varepsilon)$, *where* $\varepsilon \in \mathbb{R}^{\ell}_{\geq 0}$ *is a parameter vector, is said to be semi-globally practically asymptotically (SPA) stable uniformly in* $(\varepsilon_1, \ldots, \varepsilon_j)$, $j \in \{1, \ldots, \ell\}$, *if there exists* $\beta(\cdot, \cdot) \in \mathcal{KL}$ *such that the following holds. For each pair of strictly positive real numbers* (Δ, ν), *there exist real numbers* $\varepsilon_k^* = \varepsilon_k^*(\Delta, \nu) > 0$, $k \in \{1, 2, \ldots, j$, *and for each fixed* $\varepsilon_k \in (0, \varepsilon_k^*)$, $k \in \{1, 2, \ldots, j$, *there exists* $\varepsilon_i = \varepsilon_i(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{i-1}, \Delta, \nu)$, *with* $i = j+1, j+2, \ldots, \ell$, *such that the solutions of* $\dot{x} = f(t, x, \varepsilon)$ *with the so constructed parameters* $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_\ell)$ *satisfy*

$$|x(t)| \leq \beta(|x(t_0)|, (\varepsilon_1, \ldots, \varepsilon_\ell)(t - t_0)) + \nu, \tag{14}$$

for all $t \geq t_0 \geq 0$ *and* $|x(t_0)| \leq \Delta$. *If we have that* $j = \ell$, *then we say that the system is SPA stable, uniformly in* ε.

Observe that Definitions 5 - 7 can be restated in terms of semi-global practical stability for systems of the form $\dot{x} = f(t, x, u, \varepsilon)$, where $\varepsilon > 0$ is a small parameter.

Chapter 1
Introduction

1.1 Motivation and scope

MEASURING all the variables of interest of a dynamical system might be infeasible or prohibitively expensive. When state variables are required and not measured, they need to be estimated by using a virtual mathematical tool called observer or estimator. Observers are the solution to the estimation problem where the measured variables delivered by sensors are used to obtain the estimates of unmeasured or hidden variables of the system. In general, the structure that describes the link between the unknown and the measured variables in the estimation problem is made of three components:

- A dynamical model that describes the evolution of the variables of the plant.
- An output model that relates the state and the measured signal.
- A virtual dynamical model, called observer or estimator, that relates the state estimate, the hidden variables and the measured output.

The problem of estimating the state variables, or the so-called observer design, has been of central importance in control theory to improve product quality, enable process control, achieve fault diagnosis, and so on. This problem has been addressed from two different perspectives in the linear and nonlinear case: deterministic and stochastic. There exists a well-known robust estimation framework for linear systems based on the Luenberger observer for the deterministic case, see [16, 20, 37, 41, 42, 71, 85, 86, 98]. Regarding the case when stochastic differential/difference equations model the system there is a solid linear estimation framework based on the Kalman Filter, see [41, 62–64, 117].

Estimation theory of linear systems can be applied to nonlinear plants after linearisation of the model of the system. However, in these cases, the Luenberger observers are only valid locally and only work well when the system evolves close to the equilibrium point considered for the linearisation. On the other hand, the nonlinear version of the Kalman filter, the so-called Extended Kalman filter, iteratively linearises the state and measurement equations by computing their Jacobian matrices and evaluating these matrices at the current estimate. This linearisation may produce highly unstable filters if the restrictions of the local linearity are violated when a 'bad' estimate is used for the linearisation [62]. Furthermore, the derivation of the Jacobian matrices needed for the periodical linearisation may potentially be non-trivial and lead to implementation issues. Another well-known drawback of the Extended Kalman filter is that it does not consider the approximation errors arising from the linearisation of the state and output equations.

To overcome the local properties and the issues arising when linear estimation theory is used on nonlinear systems, a wide variety of nonlinear observers have been developed [7, 9, 12, 16, 23, 26, 43, 66, 68, 69, 71, 76, 78, 98, 103]. The estimation problem of nonlinear systems has been addressed by different approaches and perspectives with particular methodological objectives that apply to a range of classes of nonlinear systems. Although many natural and engineering systems exhibit models with multiple time scales, observers for general nonlinear systems exhibiting a time-scale separation are missing. The time scale separation arises when some variables evolve in time much faster than the rest. The analysis of multiple time-scale systems is challenging as classical methods lead to ill-posed problems. Moreover, model-based observer design for these systems is a complicated problem as the time scale separation may lead to ill-conditioned observer gains and undesired convergence properties [31, 68, 77]. If the system has two time-scales, we say that we are dealing with a system in a singularly perturbed structure [52, 56, 74, 121]. This work focuses on two time-scales systems as they are standard in a wide range of applications. For instance, consider the simplified model of a biological process with two species given by

$$\dot{x}_1 = x_2 - x_1, \tag{1.1a}$$

$$\dot{x}_2 = -x_2^3 + 2x_2 - x_1 - 1, \tag{1.1b}$$

$$\varepsilon\dot{z} = x_1 - z, \tag{1.1c}$$

$$y = z, \tag{1.1d}$$

where x_1 and x_2 represent the reaction rates of the each specie and z is the state of a sensor with a fast linear dynamics. In this case, the singular perturbation parameter

$\varepsilon > 0$ represents the time-constant of the sensor. We use this example as a motivation for the first and second part of this thesis. Even though this thesis focusses on systems with two time-scales, systems exhibiting multiple time scales are analysed using similar techniques [52, 121].

A typical academic example of a singularly perturbed system is the Van der Pol oscillator with an RL circuit where the oscillator represents a prototype vibrating actuator, and the RL circuit acts as an elementary linear plant [118]. The time scale separation appears because the Van der Pol oscillations are faster compared to the dynamics of the RL circuit. Another example is the class of systems with fast sensors and/or actuators in which the dynamics of the sensor/actuator are much faster than the dynamics of the plant [75]. Most electromechanical systems exhibit multiple time scales because, in general, the electrical variables change much faster than the mechanical ones [74]. Chemical processes and systems exhibit multiple time scales behaviours too; for instance, the selective catalytic reduction systems where the temperature variations and the change of the mass of the substrate evolve in two different time scales [120]. Nonlinear network systems such as swarms of robotic vehicles and animal aggregations with areas of internally dense and externally sparse interconnections exhibit a time-scale separation [19]. Other examples are power electronic systems [73], electrochemical systems [112], biological processes [47, 68], chemical processes [77] and so on.

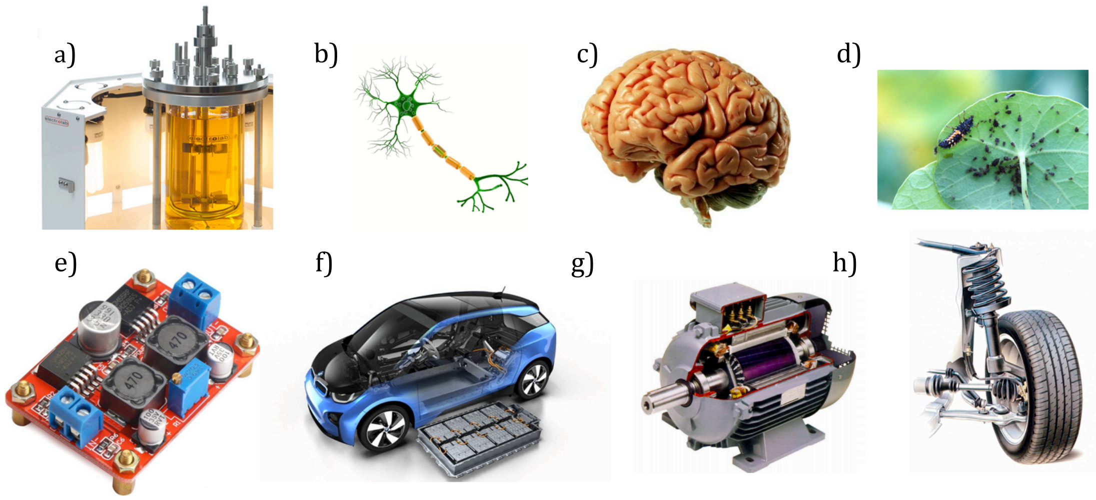

Figure 1.1: Natural and engineered systems that exhibit a time scale separation: a) continuously stirred tank bioreactor, b) a single neuron, c) human brain, d) larvae prey-predator system, e) DC-DC converter, f) lithium-ion batteries, g) electrical motor, h) suspension system.

Some examples of natural and engineered systems with variables evolving in different time-scales are displayed in Figure 1.1. For instance, the stirred tank bioreactor in a) is an instrumented system which usually has a sensor with fast linear dynamics to measure the cell-mass concentration of the system [40]. The single neuron model in b) exhibits a time-scale separation as the three main processes of the system: stimuli, reaction and relaxation happen at different time rates [47]. The human brain in c) also has multiple time-scales as the cortex activity happens in different time-scales depending of the region of the cortex [99]. A prey-predator system where the immature and mature stages of the prey are considered in the model also has a time-scale separation as the dynamics of the density of immature preys is a slow process [97]. The prey-predator system is depicted by d).

In the DC-DC converter in e), the inductor currents are faster variables than the capacitor voltages so that this sort of power electronic systems can be studied as a singularly perturbed system [73]. The picture in f) refers to lithium-ion batteries which exhibit multiple time-scales from different perspectives; for example, the internal model of a lithium-ion battery exhibits two or more time-scales depending on its desired complexity. The study of the state of health and state of charge of a lithium-ion battery also leads to a two time-scale separation [54, 72, 130, 131]. The electrical motor in g) represents the class of electromechanical systems in which the electrical variables are much faster than the mechanical ones [70, 75]. The suspension system in h) is a mechanical system that has a model that can be written in a singularly perturbed structure in which the time scale separation arises from the different magnitudes of the natural frequency of the car's body and the natural frequency of the tire [70]. Observe that analysing the aging of the spring of a suspension system also leads to a system with two time-scales.

Many works have tackled the observer design problem of linear singularly perturbed systems. However, the estimation problem for general nonlinear systems with two time-scales has not been fully addressed as current existing results only apply to particular cases. We present an overview of literature of linear and nonlinear estimation for singularly perturbed systems in Section 1.3. To the best of our knowledge, the estimation of the slow state, as well as the estimation of the slow and fast state of general nonlinear singularly perturbed systems, are open problems. The existing results on observer design for the slow variables of systems with two-time scales are very restrictive. Hence, a general framework for the estimation of the slow state of nonlinear singularly perturbed systems is needed. In this thesis, we address the slow state estimation of systems with two time-scales with a generality that has not been reported before. This generality is understood as the feature of our statements of covering many classes of nonlinear singularly perturbed systems and observers of general dimension with a single proof. We estimate the slow variables of a singularly perturbed system by using the problem setting depicted by Figure 1.2. A detailed explanation of the problem is discussed in Section 1.2, and the contributions of the thesis regarding this topic are described in Section 1.4.

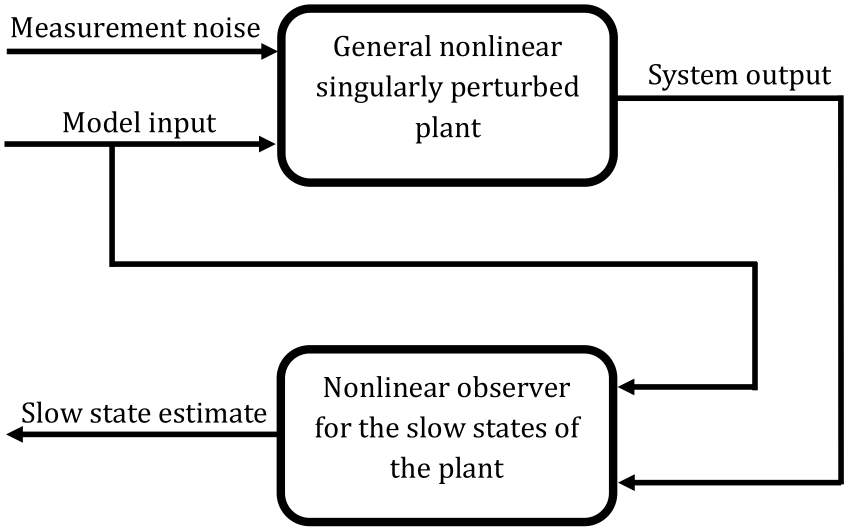

Figure 1.2: General block diagram of the slow state estimation problem.

In this thesis, we also address the parameter and state estimation problem of nonlinear systems with unknown slowly time-varying parameters. We present an estimation technique that is a generalisation of the multi-observer approach in [25] to cover the case of parameter and state estimation of systems with slowly time-varying parameters under supervisory framework depicted by Figure 1.3. We propose a new dynamic sampling policy for the multi-observer approach which is able to deal with parameters that are slowly changing. We present convergence results that guarantees that the parameter and state estimation errors have ultimate bounds that can be made arbitrarily small if the parameter moves sufficiently slow and if the observer is carefully tuned. We have addressed this problem since it is natural to the singular perturbations framework as the slow state can be regarded as a slowly time-varying parameter to the fast dynamics. We further explain the contributions on parameter and state estimation in Section 1.4.

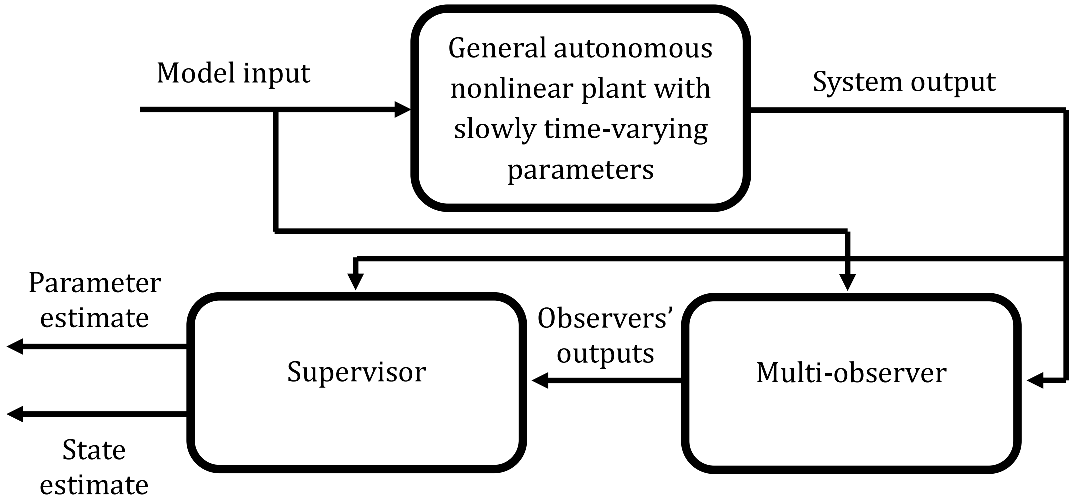

Figure 1.3: General block diagram of the parameter and state estimation of nonlinear systems with slowly time-varying parameters.

While the significance of the first set of results of this thesis lies on their generality rather than on the approach used in the proofs, the results on the multi-observer technique give the novelty factor to this work because of the technical challenges that have been addressed.

1.2 A brief introduction to singular perturbations

The perturbations framework is a compendium of methods for the systematic analysis of the global performance of solutions to differential and difference equations that exhibit a multiple time-scale structure [52]. We typically study systems with two-time scales by using an asymptotic method called singular perturbations approach [70, 74]. The general set up for singular perturbations theory consists of identifying a small parameter, usually denoted by $\varepsilon > 0$, such that when $\varepsilon = 0$, the problem becomes simpler and solvable since part of the model degenerates into an algebraic equation. Such algebraic equation allows to analyse linear/nonlinear systems exhibiting two time-scales by using lower dimensional models that approximate the performance of the original system. Hence, the global solution to the given problem is studied by performing a local analysis around $\varepsilon = 0$. In summary, singular perturbations methods are techniques that deliver approximated solutions of a nonlinear system such that the error between the real solution and the approximation is small, in some norm, for small values of $\varepsilon > 0$. Hence, by taking advantage of the time scale separation, we can avoid the ill-posed problems produced by the singular perturbation parameter on systems with two time-scales [70, 74, 75].

Below, we present a brief introduction to singular perturbations techniques. Consider the following autonomous singular perturbed system in standard from without inputs

$$\dot{x} = f(x, z), \tag{1.2a}$$

$$\varepsilon \dot{z} = g(x, z), \tag{1.2b}$$

$$y = h(x, z, w), \tag{1.2c}$$

where $x \in \mathbb{R}^n$ represents the slow state of the system, $z \in \mathbb{R}^m$ is the the vector of states associated with the fast dynamics, $y \in \mathbb{R}^p$ is the measured output of the system, $w \in \mathbb{R}^q$ is the measurement noise disturbing the output and $\varepsilon > 0$ is the singular perturbation parameter that captures the time scale separation.

Remark 1.1. *In the context of this thesis,* measurement noise *refers to an external disturbance affecting the output of the system. Such a disturbance can be high or low frequency since we only consider its amplitude to analyse its effect on the observer design problem. Although the study of the stochastic properties of the* measurement noise *is an important problem in its own right, it is out of the scope of this thesis.*

Remark 1.2. *The initial conditions of all of the nonlinear systems considered here are assumed to be unknown. The only known information about the initial conditions is the set where they belong to. We specify such a set for all of our results. The initialization of the nonlinear observers considered here is an interesting problem that is left for further research.*

The first challenge to address in the singular perturbations approach is the construction of a model for the physical system that agrees with a singularly perturbed system in the standard form (1.2). This may represent an issue since some systems that possess two time-scale properties may have non-standard representations where all variables may possess boundary layers and converge to quasi-steady-states [Section 1.6, 75]. Hence, a non-singular change of coordinates to transform the system into the standard form (1.2) would be needed. Another potential problem is the selection of the appropriate parameter to be considered as small in model (1.2) as it is not always clear how to perform such selection. In general, the understanding and knowledge of the physical components of the processes or systems are powerful tools to define an adequate perturbation parameter [70, 75].

The study of nonlinear singularly perturbed systems is complicated due to the nonlinearities and to the time scale separation; however, we can take advantage of this time scale separation to reduce the complexity of the problem. Hence, by using the fact that the states evolve in different time scales, we approximate the performance of the slow variables through the so-called reduced (slow) order model, while the fast transient is described by the discrepancy between the response of the reduced model and that of the full model [70]. The fast performance is studied through the so-called boundary layer system. The analysis of the full-system is carried out by studying the properties of the lower dimensional models and the effects of their interconnection.

By following singular perturbations techniques, we set $\varepsilon = 0$ in (1.2) such that a fundamental and abrupt change in the dynamics properties of the system is obtained. Observe that (1.2b) degenerates into an algebraic or transcendental equation

$$0 = g(x, z). \tag{1.3}$$

It is assumed that there exists an isolated solution to (1.3) given by

$$z = H(x) \quad \forall \quad x \in \mathbb{R}^n. \tag{1.4}$$

Then, setting $\varepsilon = 0$ represents the restriction of the solutions of the system (1.2) to the slow manifold given by (1.4). Hence, we can ensure a well-defined n-dimensional reduced model.

Remark 1.3. *The isolated solution (1.4) is an approximation of the slow manifold for* $0 < \varepsilon \ll 1$*. We abuse the terminology in this thesis by calling slow manifold to its approximation obtained when setting* $\varepsilon = 0$.

We now substitute the isolated solution (1.4) into (1.2a) so that we obtain the following reduced (slow) order model

$$\dot{x} = f(x, H(x)), \tag{1.5a}$$

$$y_s = h(x, H(x), w). \tag{1.5b}$$

Remark 1.4. *Let $(x(t), z(t))$ denote the solution to (1.2a) - (1.2b) for a given initial condition $(x(0), z(0))$ which is dependent on $\varepsilon > 0$. Let $\bar{x}(0)$ be an approximation of $x(0)$ at $\varepsilon = 0$ such that $\bar{x}(t)$ is the solution to (1.5a). Define the quasi-steady-state of $z(t)$ when $x(t) = \bar{x}(t)$ as $\bar{z}(t) := H(\bar{x}(t))$. Since $\dot{z} = g/\varepsilon$, the speed of z can be high when ε is small and whenever $g \neq 0$. Hence, we have made the transient of z instantaneous when setting $\varepsilon = 0$ in (1.2). Note that appropriate stability conditions must be satisfied to ensure that $z(t)$ will converge to its quasi-steady-state $\bar{z}(t)$, which represents an equilibrium of (1.2b). Such conditions will be defined below. We know from [Theorem 11.2, 70] that for a sufficiently small $\varepsilon > 0$ the following conditions hold: $x(t) - \bar{x}(t) = O(\varepsilon)$ and $z(t) - \bar{z}(t) = O(\varepsilon)$. Therefore, as the approximated solutions remain arbitrarily close to the real ones after the boundary-layer transient, we will use x and z instead of $\bar{x}$ and $\bar{z}$ when referring to the approximated variables. This use of the notation is standard in the literature on singular perturbations [Chapter 7, 75], [33, 70, 118].*

Singular perturbations theory suggests us to perform a change of variables to analyse the behaviour of the fast state of the system (1.2). Hence, let consider the change of variables

$$\xi = z - H(x), \tag{1.6}$$

which shifts the quasi-steady-state of $z \in \mathbb{R}^m$ to the origin. Observe that, in the new variables (x, ξ), the full system (1.2) becomes

$$\begin{aligned} \dot{x} &= f(x, \xi + H(x)), && (1.7a) \\ \varepsilon \dot{\xi} &= g(x, \xi + H(x)) - \varepsilon \frac{\partial H}{\partial x} f(x, \xi + H(x)), && (1.7b) \\ y &= h(x, \xi + H(x), w). && (1.7c) \end{aligned}$$

To introduce the boundary layer system, we first define the fast-time variable $\tau := t/\varepsilon$. Hence, the system (1.7) in the τ-time scale is given by

$$\begin{aligned} \frac{dx}{d\tau} &= \varepsilon f(x, \xi + H(x)), && (1.8a) \\ \frac{d\xi}{d\tau} &= g(x, \xi + H(x)) - \varepsilon \frac{\partial H}{\partial x} f(x, \xi + H(x)), && (1.8b) \\ y &= h(x, \xi + H(x), w). && (1.8c) \end{aligned}$$

Setting $\varepsilon = 0$ in the system (1.8) leads to the so-called boundary layer system

$$\frac{dx}{d\tau} = 0, \tag{1.9a}$$
$$\frac{d\xi}{d\tau} = g(x, \xi + H(x)), \tag{1.9b}$$
$$y_f = h(x, \xi + H(x), w), \tag{1.9c}$$

which has equilibrium at $\xi = 0$. The slow state x in (1.9) is frozen to its initial value and is seen as a fixed parameter for the fast dynamics. Since the slow variables eventually move away from their initial conditions, an asymptotic stability of this equilibrium is needed to justify the model reduction and to be able to analyse the full system (1.7) via the reduced system (1.5) and the boundary layer system (1.9).

By using the lower dimensional systems (1.5) and (1.9), the analysis and design tasks become a more manageable problem. Systems with two-time scales are notoriously hard to deal within the context of state estimation since the time scale separation may lead to ill-conditioned gains that inherently causes complications in the observer design. When dealing with the estimation problem, we work with the reduced order and boundary layer systems to overcome undesired convergence properties of the estimation error when the observer is designed for the full plant [88]. In Part I and II of this thesis, we concentrate on estimating the slow states of the plant by using an observer synthesized for the reduced model. We put our attention on this problem as several systems and processes with "slow" dynamics are instrumented with sensors or actuators that exhibit "fast" dynamics. For instance, reactor networks and some classes of bioprocess [68]. We use the standard methodology on state estimation of linear/nonlinear singularly perturbed systems [24, 38, 58, 59, 68, 126]. This methodology is summarised as follows

1. We approximate the full plant via the lower dimensional systems: reduced order system and boundary layer system.
2. We design an observer for the reduced model while the fast variables are neglected.
3. We implement the observer on the full system.

Since we implement the observer on the original plant and the fast variables of the system are not considered during the observer design, we analyse the robustness of the observer to singular perturbations. Moreover, we study the estimator robustness to measurement noise affecting the output of the system.

Consider the system (1.5) and design an observer of the following form

$$\dot{\hat{x}} = f_o(\hat{x}, y_s), \tag{1.10}$$

where $\hat{x} \in \mathbb{R}^n$ is the state of the observer and an estimate of $x \in \mathbb{R}^n$. To be able to design an observer for the reduced order system (1.5), we need to guarantee that (1.5) has bounded solutions since the estimation problem of unbounded systems is a hard task to address. Hence, assume that the solutions to (1.5) are globally bounded so that it is possible to design the observer (1.10). Note that as the system initial conditions are unknown, the problem of choosing the observer initial conditions is an interesting problem when stating semi-global results. Methods for the appropriate choice of the observer initial conditions are out of the scope of this thesis and are left for future research.

We now define the state estimation error as $e = x - \hat{x}$ so that the error dynamics are given by

$$\dot{e} = f(x, H(x)) - f_o(\hat{x}, y_s). \tag{1.11}$$

To guarantee an appropriate performance of the observer (1.10) when implemented on the plant (1.7), we need an observer for which the estimation error has an appropriate performance when used on the reduced system (1.5). Hence, we choose an observer for which the error dynamics (1.11) satisfy certain stability property; for instance, ISS stability with respect to the measurement noise. Note that the estimation of the slow states via an observer synthesised for the reduced order system has been used before in linear/nonlinear singularly perturbed systems [24,38,58,59,68,126]. Since the observer must be implemented on the singularly perturbed plant (1.7) to estimate the slow state, we compute the slow state estimation error as follows

$$\dot{e} = f(x, \xi + H(x)) - f_o(\hat{x}, y). \tag{1.12}$$

Hence, we need to investigate the robustness of the observer with respect to singular perturbations as the observer is synthesised without considering the fast variables. The block diagram in Figure 1.4 summarise the estimation problem of the slow states of the plant described above where the observer is synthesised for the reduced system (1.5) and used to estimate the slow state of the plant. Figure 1.4 depicts a simplified version of the problem setting we address in Chapters 2 and 4 of this thesis.

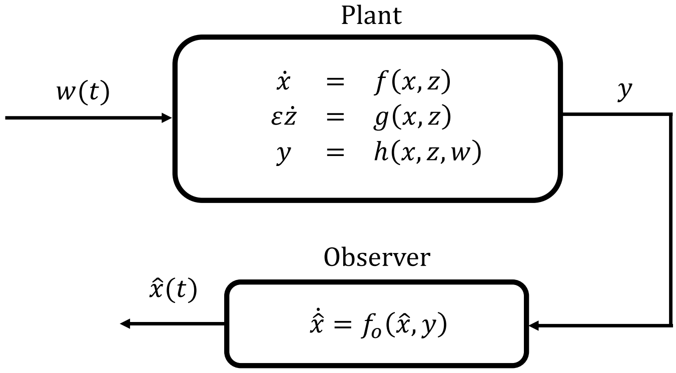

Figure 1.4: Summary of the slow state estimation problem.

Consider the motivational example introduced in Section 1.1 which is defined by equation (1.1). Note that, by setting $\varepsilon = 0$, one can obtain the reduced order system with the following model

$$\dot{x}_1 = x_2 - x_1, \tag{1.13a}$$
$$\dot{x}_2 = -x_2^3 + 2x_2 - x_1 - 1, \tag{1.13b}$$
$$y_s = x_1. \tag{1.13c}$$

Then, by using the change of variables $\xi = z - x_1$ and introducing the fast time-scale $\tau = t/\varepsilon$, it follows that the boundary layer system is given by

$$\frac{d\xi}{d\tau} = -\xi, \tag{1.14a}$$
$$y_f = \xi + x_1(0). \tag{1.14b}$$

Observe that the reduced order system (1.13) can be written in the following form

$$\dot{x} = Ax + G\gamma(Fx) + \sigma(y_s, u), \tag{1.15}$$

where $x = [x_1, x_2]^T$ and

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \sigma(y_s, u) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Then, we can design a circle criterion observer [9] with the following form

$$\dot{\hat{x}} = A\hat{x} + L(C\hat{x} - y_s) + G\gamma(F\hat{x} - K(C\hat{x} - y_s)) + \sigma(y_s, u), \tag{1.16}$$

where $\hat{x} \in \mathbb{R}^2$ is the state of the observer and an estimate of $x \in \mathbb{R}^2$, and the gain matrices are $K = -2.49$ and $L = [-3.24, -13.52]^T$. By following the approach described above, we implement the observer (1.16) on the original plant (1.1) to estimate the slow variables of the system. We show in Figure 1.5 the performance of the observer for different values of the perturbation parameter $\varepsilon > 0$. It can be seen that the estimation error has better convergence properties when the perturbation parameter is smaller.

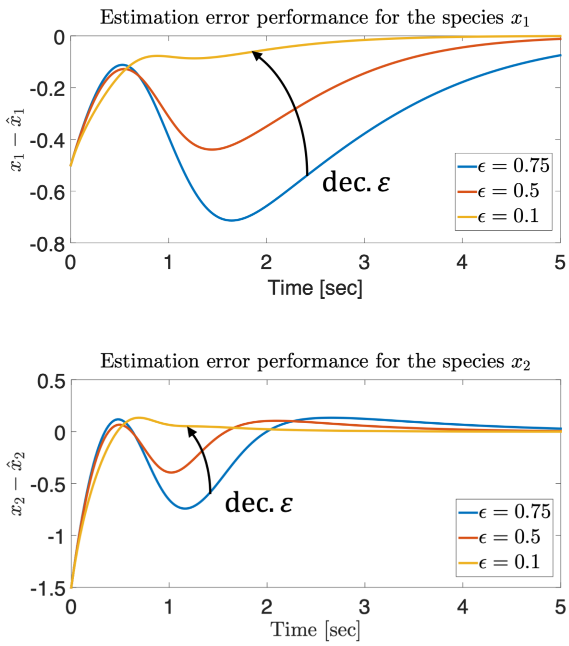

Figure 1.5: Simulation results for the motivational example.

1.3 Overview of literature

Many nonlinear observer design methods can be found in the literature. Different approaches and perspectives with particular methodological objectives have been developed to deal with nonlinearities in processes and systems. For instance, the authors of [76] propose a geometric observer design method in which they obtain linear estimation error dynamics. A less conservative methodology is presented in [69] where the set of conditions in the theoretical body, especially regarding linearisation, are much less restrictive than in other works. Another nonlinear observer is the circle criterion observer which provides globally convergent estimates for a class of nonlinear models, see [9]. This observer can be designed and implemented on a class of systems in which the nonlinearities satisfy a monotone growth property [7, 9, 43]. A common class of observers are the so-called high-gain observers in which high-gain linear terms or geometric transformations counter the nonlinearities of the plant, see [71].

Other nonlinear observers include the nonlinear versions of the Kalman filter: the Extended Kalman filter [117] and the Unscented Kalman filter [62]. The Extended Kalman filter solve the nonlinear estimation problem by iteratively linearinsing the nonlinearities of the system's model and using the traditional linear Kalman filter. This technique presents some issues as the linearisation may produce highly unstable filters. Hence, the Unscented Kalman filter was developed to obtain the equivalent performance of the Kalman filter in nonlinear systems without the linearisation steps required by the Extended Kalman filter. The main element of the Unscented Kalman filter is the unscented transformation which uses a set of appropriately chosen weighted points to parametrise the means and covariances of probability distributions [62]. This filter does not require to compute the Jacobians matrices so that has a larger scope of applicability. The Extended Kalman and the Unscented Kalman filters only provide local estimates of the state. There are other many nonlinear observers for particular applications; for instance, observers for permanent magnet synchronous motors [103], observers designed via dynamic extension [67] and so on [3, 8, 13, 17, 18, 84, 119].

In this thesis, we prove that a number of the nonlinear observers can be used to estimate the slow states of a singularly perturbed system by synthesising an observer based on the reduced system (1.5) and implementing it on the full plant. Although the design framework for the slow state estimation generated in this thesis do not cover all the aforementioned nonlinear observers, we demonstrate in Chapters 3 and 5 that a number of observers fit such a design framework.

1.3.1 Observer design for singularly perturbed systems

Regarding observer design for singularly perturbed systems, there exists a robust estimation framework for linear singularly perturbed systems where the observer design problem has been tackled from different perspectives [39, 44, 45, 53, 59, 65, 79, 81, 82, 93, 100, 102, 113, 124, 127, 129]. For instance, a composite observer to estimate the full state of a class of time-varying linear singularly perturbed systems is presented in [101]. The composite observer is constructed by using an estimator designed for the reduced model and an observer synthesised for the boundary layer system. Another work is presented in [58] where the authors propose a technique to estimate only the slow state of a linear singularly perturbed plant with equilibrium manifolds. The observer design is done based on the reduced model of the plant. The stochastic linear case has been covered too; for instance, an observer design method that deals with linear singularly perturbed systems with uncertain perturbation parameters is presented [111]. In the stochastic case, problems arise because the white noise is faster than the fast dynamics of the boundary layer [44, 49, 111]. Whilst there are several results on observer design for linear singularly perturbed systems, the nonlinear case has not been fully explored.

As far as we are aware, the most recent results on observer design for linear singularly perturbed systems are reported in [125–127]. The authors address the estimation problem of linear systems with slow and fast modes by introducing independent slow and fast observers. Besides providing pure-slow and pure-fast observers, the proposed observer design methodology produces estimates with very high accuracy of $O(\varepsilon^i)$, for $i \in \{2, 3, \ldots\}$. This feature of the observers is significant since most of existing methods for estimation of linear singularly perturbed systems only deliver an $O(\varepsilon)$ accuracy. The observer design results are then used to construct independent observer-based controller by using a two-stage feedback design technique for the slow and fast subsystems. The authors show that the typical ill-conditioning problem of singularly perturbed systems is avoided under the proposed design approach. They analyse different cases depending on the measured state space variable. Results in [125–127] have significantly contributed to the knowledge regarding linear estimation of systems with two time-scales.

The observer design literature for nonlinear singularly perturbed systems is not as extensive as in the linear case. Although multiple time-scale estimation has been used in some nonlinear applications [28–30,32,34,54,58,107,108,130,131], there is no general mathematical framework on the stability of the estimation error covering a large class of plants and observers since existing results cover specific cases. For instance, there is a solid framework on feedback control for singularly perturbed systems where nonlinear observers are used to design feedback controllers [28–30, 32, 34]. The authors of [32] study the output feedback control of two time-scale hyperbolic PDE systems by using two distributed state observers which are synthesised for the fast and slow subsystems. Then, these observers are used together with a distributed state feedback controller to guarantee closed-loop stability of the system and enforce output tracking. The main assumption of this work is that the perturbation parameter is sufficiently small such that the time scale septation is sufficiently large.

The author of [30] addresses the problem of synthesising a robust output feedback controller for a class of nonlinear singularly perturbed systems with uncertain variables. It is assumed that the system has an asymptotically stable boundary layer system and a reduced order system which is input/output linearisable and possesses input-to-state inverse dynamics. The problem is tackled by using a combination of a high-gain observer and a robust state feedback controller synthesised via the Lyapunov's direct method. Even though results in [28–30, 32, 34] are highly useful for stabilisation and output tracking, they only apply to specific classes of systems and do not possess the generality to be applicable to many problems. Moreover, their main objective is not the estimation problem by itself.

There are few other works on estimation for nonlinear systems with two time-scales. For instance, the authors of [130] propose an algorithm to estimate the internal states of a lithium-ion battery to enable real-time monitoring and control of the state of charge and state of health of the battery. The author uses a multi-time scale approach since the state of charge evolves faster than the state of health. The estimation of the state is done by using an Extended Kalman filter [117]. Moreover, it is stated in [130] that the algorithm can be implemented by using a Unscented Kalman filter [62]. The works presented in [54, 131] are based on the Extended Kalman Filter too. In [131], two filters with different time-scales are combined for the state of health and state of charge estimation in lithium-ion batteries. The state of charge is estimated in real-time, and the state of health is updated off-line. The authors of [54] address the same problem by proposing a multi-scale framework with the Extended Kalman filter which when applied to the battery system can be regarded as a hybrid of Coulomb counting and adaptive filtering techniques. Observers to estimate the full state of a class of mechanical singularly perturbed systems are presented in [107, 108]. Both works propose a Lyapunov-based observer design; however, their results only apply to spring-mass-damper systems since the observer is constructed by considering specific characteristics of the model.

Despite the fact that the nonlinear estimation problem for systems exhibiting two time-scales has been studied in some examples as the ones mentioned above, those results only apply to particular cases. To the best of our knowledge, there are few rigorous results on nonlinear estimation for general singularly perturbed systems. For instance, a systematic natural observer design framework for vector second-order systems in the presence of multiple time scales is presented in [38]. Second-order mechanical systems with fast unmodelled sensor dynamics were considered. Further results for deterministic nonlinear systems are presented in [68] where the authors deal with the estimation of the slow state based on the reduced order model. The study is focused on a specific class of plants with fast linear dynamics and a particular nonlinear observer with linear estimation error dynamics. The fast variables of the system are not estimated, and they are neglected during the observer design. Then, the authors study the effect of the fast variables on the convergence properties of the estimation error. Note that these results are restrictive in the sense of applying to a particular class of systems and a specific nonlinear observer.

Another work on the estimation of the slow state of a system with two time-scales is presented in [24]. The authors introduce a sliding mode observer for estimating the slow variables of a class of nonlinear globally Lipschitz plants. In this context, the estimation of the slow and fast state of nonlinear globally Lipschitz systems via a Luenberger-type observer is reported in [123]. The authors consider a limited class of nonlinear systems with global Lipschitz properties. The observer design is carried out in terms of an LMI condition which is independent of the perturbation parameter and which guarantees exponential stability of the error dynamics. In [36], we can found estimation results on the Extended Kalman Filter for a class of singularly perturbed stochastic nonlinear systems.

In summary, we can identify there is a lack of general results for the state estimation of nonlinear singularly perturbed systems. Although the problem has been addressed for some particular problems, we are not aware of existing results with the enough generality to cover large classes of nonlinear systems and nonlinear observers of general dimension. In fact, we can identify two important research gaps in the literature that need to be filled,

- A general estimation framework for slow state estimation of nonlinear singularly perturbed systems.
- An estimation technique to estimate both slow and fast states of nonlinear singularly perturbed systems.

We now make a brief overview of an important rigorous mathematical result on slow state estimation. We concentrate on the result in [68] as it is the most closely related to the first and second parts of this thesis. Then, we revise the required literature for the parameter and state estimation of nonlinear systems with unknown slowly time-varying parameters which is the problem addressed in the third part of the thesis.

An estimation result for nonlinear singularly perturbed systems

This thesis focuses on the deterministic estimation problem of nonlinear systems with two time-scales. Hence, we revise those results in [68] where the authors work with the standard approach for the linear/nonlinear observer design for the estimation of the slow variables of autonomous singularly perturbed systems. They consider a specific class of nonlinear plants that covers processes and systems exhibiting fast linear dynamics. The slow part of the state is reconstructed through a state observer which is designed based on the reduced model. Then, the dynamic behaviour of the estimation error is analysed and mathematically characterised when the observer is implemented on the full system. We revise results in [68] as they represent the starting point of the first and second parts of this thesis. The class of systems considered in [68] has the following standard singularly perturbed form

$$\dot{x} = f(x, z), \tag{1.17a}$$

$$\varepsilon\dot{z} = M_1 x + M_2 z, \tag{1.17b}$$

$$y = C_1 x + C_2 z, \tag{1.17c}$$

where $x \in X \subset \mathbb{R}^n$ is the slow state of the system, $z \in Z \subset \mathbb{R}^m$ is the fast part of the state and X, Z are compact sets containing the origin, $\varepsilon > 0$ is the singular perturbation parameter which is assumed to be small, and $y \in \mathbb{R}$ is the measured output of the system. It is assumed that $f(x, z)$ is a real analytic vector function defined on $X \times Z$, and M_1, M_2, C_1, C_2 are constant matrices of appropriate dimensions with M_2 being non-singular. The class of systems represented by (1.17) captures a broad class of interesting cases such as instrumented processes, biological processes, reactor networks, and so on [68].

It is assumed that the origin $(x, z) = (0, 0)$ is an equilibrium point for (1.17) with $f(0, 0) = 0$. By setting $\varepsilon = 0$, the system is restricted to the slow manifold $M_1 x + M_2 z = 0$ such that the reduced system is obtained

$$\dot{x} = f(x, -M_2^{-1} M_1 x), \tag{1.18a}$$

$$y = (C_1 - C_2 M_2^{-1} M_1) x. \tag{1.18b}$$

By considering the reduced model (1.18), a simplified design of an appropriate nonlinear observer is done to estimate the slow variables of the original plant (1.17). So, consider the nonlinear observer introduced in [69], which has the following form

$$\dot{\hat{x}} = f(\hat{x}, -M_2^{-1}M_1\hat{x}) + L(\hat{x})(y - \hat{y}), \tag{1.19a}$$

$$\hat{y} = (C_1 - C_2M_2^{-1}M_1)\hat{x}, \tag{1.19b}$$

where $\hat{x} \in \mathbb{R}^n$ is the state of the observer and an estimate of $x \in \mathbb{R}^n$, and $\hat{y}$ is the estimated output. The above nonlinear observer has a state-dependent gain $L(x)$, which can be computed as follows

$$L(x) = \left[\frac{\partial T}{\partial x}(x)\right]^{-1} B, \tag{1.20}$$

where $w = T(\bar{x}) : \mathbb{R}^n \to \mathbb{R}^n$ is a solution to the following associated system of first order non-homogeneous linear partial differential equations (PDEs)

$$\frac{\partial T}{\partial x} f(x, -M_2^{-1}M_1x) = AT(x) + B(C_1 - C_2M_2^{-1}M_1)x, \tag{1.21}$$

with A, B being constant matrices of appropriate dimensions. It can be proven that the observer (1.12) with the nonlinear gain (1.20) induces a linear error dynamics given by

$$\dot{e} = Ae, \tag{1.22}$$

where the estimation error is defined as $e := T(x) - T(\hat{x})$. Therefore, if A is chosen to be Hurwitz, its eigenvalues regulate the exponential rate of decay of the estimation error to zero [68]. When the observer (1.19) designed based on the reduced system (1.18) is implemented on the original system (1.17), the presence of the fast z-dynamics leads to new estimation error dynamics. It can be shown that the estimation error dynamics of the slow observer when implemented on the original plant (1.17) is given by

$$\dot{e} = Ae - BC_2M_2^{-1}M_1x - BC_2z + \frac{\partial T}{\partial x} f(x, z) - \frac{\partial T}{\partial x} f(x, -M_2^{-1}M_1x). \tag{1.23}$$

The authors of [68] take advantage of the linear structure of the fast dynamics and use closeness of solutions results for singularly perturbed systems to show that the estimation error satisfies

$$e(t) = exp(A(t - t_0))e(t_0) + \mathbf{H}(t, \varepsilon), \tag{1.24}$$

where the observer error term $\mathbf{H}(t, \varepsilon)$ is of order $O(\varepsilon)$ for $t \in [t_0, \infty)$. Although the result in [68] is interesting on its own right, it cannot be applied to a large class of plants and nonlinear observers. Then, we aim to generalise them by considering a boarder class of systems and nonlinear observers of general dimension. We have given a brief summary of results presented in [68] since they are the most closely related results to the first and second parts of this thesis.

1.3.2 Parameter and state estimation of nonlinear systems

The parameter and state estimation problem as separate problems has been of central importance in control theory. Several approaches have dealt with the state estimation [3, 7–9, 13, 16–18, 43, 71, 84, 98, 119], and some others have addressed the the parameter estimation problem [1,55,83]. The simultaneous estimation of both parameter and state is commonly tackled by augmenting the state vector with the parameter vector. This technique transforms the parameter and state estimation problem into a state estimation task. However, augmenting the state may lead to a model with several nonlinearities which may further complicate the estimation problem. Systems with models with unknown parameters has been studied before in control literature [15, 51, 90–92, 122]. Different approaches as the supervisory control have been developed to address the problem of steering the system state to the origin. The supervisory framework is closely related to the parameter and state estimation as it uses a bank of estimators for control purposes. Nevertheless, such a framework does not provide guarantees of convergence of the parameter estimates.

The supervisory control framework for linear systems [15, 51, 90–92, 122] depicted in Figure 1.6 consists of two main units: 1) a multi-controller and 2) a supervisor that defines the switching among the controllers. The multi-controller is a family of candidate controllers parametrized by guesses of the uncertain parameters. On the other hand, the supervisor is constituted by a multi-estimator, a set of monitoring signals and a switching logic. The multi-estimator is defined as a bank of estimators that uses the input and output of the system to produce estimates of the unknown parameters and estimates of the output for each of these parameter estimates. The monitoring signals, which are functions of an appropriate norm of the output estimation errors, are used to generate a switching signal that selects the active controller at each time instant.

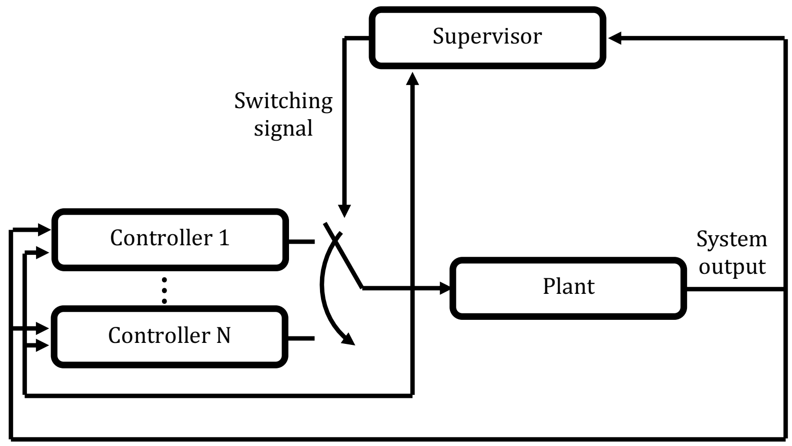

Figure 1.6: Supervisory control framework.

The supervisory control was introduced in [90] to drive and hold at a prescribed setpoint the output of a process modelled by a dynamical system with large scale uncertainty. While other techniques experimentally evaluate the performance of each candidate controller by briefly applying it to the process, the supervisory control posses a unique distinctive characteristic which is that the controller selection is made based on a continuous comparison in real time of the output estimation errors. This significant feature of the supervisory control has attract the attention of the research community so that further improvements of the switching logic have been proposed; for instance, the hysteresis-based switching logic proposed in [51].

The supervisory framework has recently motivated the parameter and state estimation of nonlinear systems by using a multi-observer. This approach was introduced in [25] where the authors propose a hybrid scheme for the parameter and state estimation of nonlinear systems inspired by the supervisory framework. In [25], the unknown parameter is assumed to be a constant vector that belongs to a known compact set. The state observers are synthesized for a finite set of nominal parameter values to generate multiple state estimates. Then, a selection criterion based on the supervisory framework chooses the estimate by using monitoring signals that consider the difference between the measured and the estimated output so that it provides state and parameter estimates at any given time instant. The multi-observer technique introduced in [25] addresses the parameter and state estimation problem as depicted by Figure 1.7.

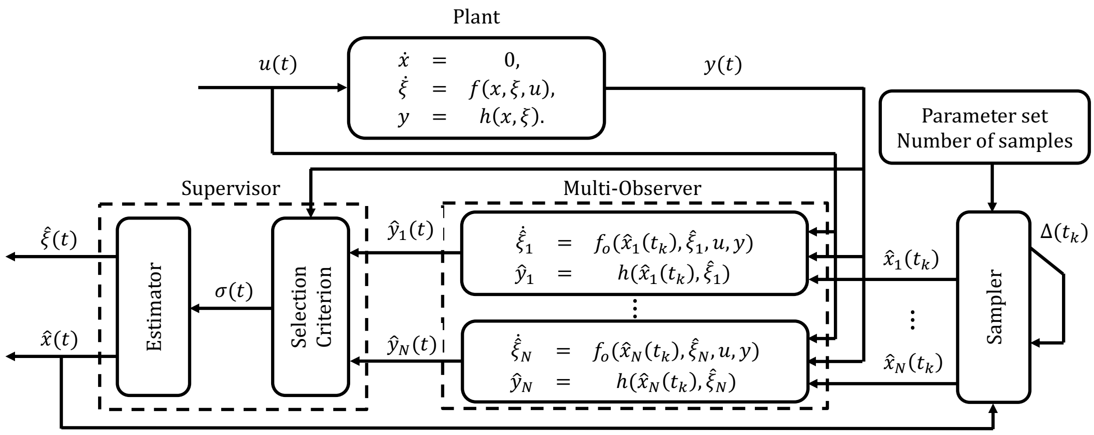

Figure 1.7: Parameter and state estimation of nonlinear systems with unknown constant parameters.

The sampler in Figure 1.7 represents a sampling strategy that generates a set of parameter sample points. The multi-observer approach requires of this set of sample points to construct the bank of observers. The authors of [25] propose two sampling policies for the generation of the parameter sample points. The first one is a static policy where a sampled set with a large number of samples is generated at the start of the algorithm and remain the same for all time. The second policy is a dynamic sampling policy where the parameter samples are periodically updated by using a zoom-in procedure to provide the same accuracy as with the static policy while using a smaller number of observers. Note that the sampler in the dynamic sampling policy uses the last parameter estimate to generate the new samples at each iteration. The number of observers determines the size of the neighbourhood around zero where the parameter and state estimation errors converge. Hence, the estimation errors can be made as small as desired by increasing the number of observers.

The multi-observer approach seems natural to the singular perturbations framework as the slow state can be regarded as a fixed parameter to the boundary layer system. In fact, results in [25] deal with nonlinear plants that can be seen as the boundary layer system (1.9) of a singularly perturbed plant (1.2) where the slow state is treated as fixed parameter. Since the slow variables behave as slowly time-varying parameters to the fast part of a singularly perturbed system, here we address the parameter and state estimation problem for nonlinear systems with slowly time-varying parameters. We generate a new sampling policy for the multi-observer approach that generalises results in [25] as it can address more general problems.

A zoom-in and a zoom-out procedures are the main elements of the new sampling policy presented in here. The multi-observer approach for systems with unknown constant parameters and the dynamic sampling policy with a zoom-in procedure in [25] as well as the control strategy with zooming-in and zooming-out procedures in [80] have inspired the work presented in here. Although the authors of [80] study a different problem (stabilization with respect to external disturbances of linear systems with quantized measurements), we have found out that the main idea behind their switching control strategy is useful in our problem. Further discussion on this is presented in introduction of Chapter 6.

1.4 Outline of the thesis and contributions

This thesis focusses on the convergence analysis of deterministic state nonlinear observers for the slow state estimation of a general class of nonlinear singularly perturbed systems. We present a general estimation framework for slow state estimation of nonlinear singularly perturbed systems, where the generality comes from the fact that these results cover large classes of nonlinear plants and observers of general dimension. This feature of the thesis sharply distinguish it from other works where results only apply to particular plants and specific observers. This thesis also introduces a novel perspective for parameter and state estimation for nonlinear systems with slowly time-varying unknown parameters that is natural to the singular perturbations framework. We now summarise our contributions and provide a brief outline for the material that is presented and developed in the subsequent chapters.

This thesis has been divided in three parts: **Part I** (Chapters 2 - 3): Slow State Estimation of Globally Lipschitz Nonlinear Singularly Perturbed Systems, **Part II** (Chapters 4 - 5): Observers of General Dimension for the Slow State Estimation of Nonlinear Singularly Perturbed Systems, and **Part III** (Chapters 6 - 7): Parameter and State Estimation of Nonlinear Systems with Slowly Time-varying Parameters.

Part I: Slow State Estimation of Globally Lipschitz Nonlinear Singularly Perturbed Systems. We develop a general estimation framework for the slow state of globally Lipschitz nonlinear singularly perturbed systems. We deal with a smaller class of nonlinear singularly perturbed systems and nonlinear observers than the plants and estimators studied in Part II. However, the strong assumptions we use in Part I lead to stronger convergence properties which are crucial for a number observers. In Chapter 2, we study the robustness of nonlinear observers with respect to singular perturbations and to measurement noise of an observer designed base on the reduced system and used to estimate the slow variables of a globally Lipschitz nonlinear singularly perturbed system. As far as we are aware, our global convergence results for nonlinear observers for singularly perturbed systems are the first ones that consider the presence of measurement noise as a disturbance to the output. These global results cover a broader class of systems than the existing results in the literature. We provide input-to-state stability and finite-gain $\mathcal{L}_2$ stability results. In Chapter 3, we demonstrate and illustrate the generality of results from Chapter 2. We show that many classes of nonlinear globally Lipschitz singularly perturbed systems satisfy the given assumptions in Chapter 2. We also demonstrate how our assumptions hold for nonlinear observers that can be designed for the reduced systems of those plants. We study four classes of systems with reduced order models for which we can design four different nonlinear observers [2,9,128]. Although we have checked that our framework covers existing results as those in [123] when the observer is used only for the estimation of the slow state, we have not presented all those cases here. Simulations results are provided in this chapter.

Part II: Observers of General Dimension for the Slow State Estimation of Nonlinear Singularly Perturbed Systems. In the first part of the thesis, we require strong assumptions that imply strong conclusions at the expense of restricting the applicability of results. Hence, by considering relaxed assumptions, we generate semi-global convergence results for nonlinear observers of general dimension used to estimate the slow state of a nonlinear singularly perturbed plant. These results are weaker convergence properties than those generated in Part I; however, they cover a larger number of plants and observers. We analyse the robustness of the observers to singular perturbations and to measurement noise. We state practical input-to-state stability results for a general class of nonlinear singularly perturbed systems and observers of general dimension when the input and the measurement noise belong to $\mathcal{L}_\infty$. We prove $\mathcal{L}_\infty \cap \mathcal{L}_2$ results when the measurement noise belongs to $\mathcal{L}_\infty \cap \mathcal{L}_2$. Furthermore, we state semi-global practical asymptotical stability of the estimation error in the absence of measurement noise. Then, Chapter 5 plays the same role as Chapter 3 in illustrating the generality of our results. We demonstrate that, for each of the classes of systems and observers presented in Chapter 5, the assumptions in Chapter 4 hold. We study the class of plants and observers considered in [68]. Furthermore, we analyse another three classes of systems and a reduced-order, a full-order and a higher-order observer. Our results cover a larger number of observers and nonlinear plants; however, we have not included all possible cases for which we can guarantee that our design framework applies.

Part III: Parameter and State Estimation of Nonlinear Systems with Slowly Time-varying Parameters. An approach for parameter and state estimation of systems with slowly time-varying parameters is proposed by using a multi-observer approach adapted from the case when the unknown parameter is constant. It is rigorously proved that the multi-observer approach provides practical convergence of both parameter and state estimates for a general class of autonomous nonlinear systems with slowly time-varying parameters. Chapter 6 introduces a novel dynamic sampling policy for the multi-observer approach which is able to deal with slowly time-varying parameters. We combine a hysteresis switching law with zoom-in and zoom-out procedures to guarantee that the parameter estimate will stay close to the slowly time-varying parameter for all time. We state convergence results that show that the parameter and state estimates are ultimately bounded. It is shown that the ultimate bound depends on two factors: 1) the number of sample points and observers, and 2) the rate of change of the slowly time-varying parameters. The new sampling policy as well as the convergence results are the main contributions of the third part of the thesis. In Chapter 7, we illustrate through simulations the applicability of results in Chapter 6. We demonstrate via simulations that our approach can be used on systems with unknown constant parameters in the presence of noisy measurements as well as on systems with discontinuous slowly time-varying parameters. Moreover, we present simulations results for the case when our technique is used to estimate the full-state of a singularly perturbed system.

Finally, Chapter 8 presents the conclusions and a discussion on the possible future research directions related to the content of this thesis. Appendices A, B an C contain the mathematical proofs of all the main results presented in here.

Part I

Slow State Estimation of Globally Lipschitz Nonlinear Singularly Perturbed Systems

Introduction to Part I

IN THIS first part of the thesis, we analyse the stability of the estimation error of the slow variables of general globally Lipschitz nonlinear singularly perturbed plants when nonlinear observers are synthesized based on the reduced (slow) system. Hence, we address the problem of slow state estimation by using the set-up shown in Figure I.1. Observe that we consider a general class of globally Lipschitz nonlinear systems where the output of the model is corrupted by measurement noise.

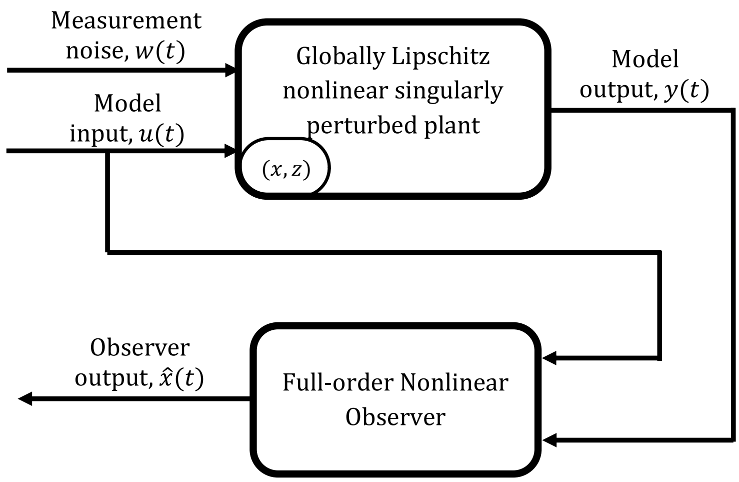

Figure I.1: General setting for the slow state estimation of globally Lipschitz singularly perturbed systems via a full order observer synthesised for the reduced (slow) system.

Here, we state strong (global) assumptions on the plant and on the convergence properties of the observer designed for the reduced order model. These global conditions lead to strong global conclusions on the stability of the slow estimation error dynamics when the observer is implemented on the original plant. These results cover a smaller class of nonlinear singularly perturbed systems and nonlinear observers than the plants and estimators studied in Part II where weaker conditions lead to weaker results. Although results in this first part and the second part of the thesis deal with similar problems, they do not imply each other as the sharper results concluded from Part I cannot be concluded from results in Part II. On the other hand, results from Part I cannot cover as many plants and observers of general dimension as results in Part II.

This part of the thesis consists of two chapters where we deliver global convergence results for the slow estimates of the plant. In Chapter 2, we focus on the theoretical developments that give a solution to the problem. We present a robustness analysis to singular perturbations and to measurement noise for general nonlinear globally Lipschitz singularly perturbed systems and full-order observers. The study is performed by using a Lyapunov approach which is combined with an ISS technique for interconnected systems. We take advantage of the fact that the estimation error is in cascade with the state of the plant. We also obtain results in terms of finite-gain $\mathcal{L}_2$ stability.

As Chapter 2 presents an estimation framework that cover a number of nonlinear plants and observers, we demonstrate the applicability of such framework in Chapter 3. We verify that at least four classes of plants and observers satisfy the stated assumptions in Chapter 2. Moreover, we present simulations results to illustrate our findings.

Chapter 2

Observers for Globally Lipschitz Nonlinear Singularly Perturbed Systems

In this chapter, we study the stability of the estimation error of full-order observers designed to estimate the slow state of globally Lipschitz nonlinear singularly perturbed plants. The observers are designed on the basis of the reduced (slow) model. We prove a strong (global) result under global Lipschitz assumptions and global exponential stability of the boundary layer dynamics. We analyse the robustness of the observers with respect to singular perturbations and with respect to measurement noise.

2.1 Introduction

ESTIMATION of the slow states of singularly perturbed systems via an observer designed for the reduced (slow) system has been studied before, see [24, 123]. However, these results deal with specific observers and specific plants with appropriate Lipschitz properties. The estimation of the slow and fast states via a Luenberger-type observer is reported in [123]. The observer design is carried out in terms of an LMI condition independent of the perturbation parameter. A sliding mode observer design for the slow states of a singularly perturbed plant is presented in [24]. Although the results in [24,123] guarantee desired convergence properties, they cannot be extended to cover a larger class of plants since their design approach takes into account the special characteristics of the considered class of systems. Hence, we use the standard singular perturbations approach to generate a general estimation framework for globally Lipschitz nonlinear singularly perturbed systems. As far as we are aware, there are no existing results in the literature addressing the slow estimation problem with the generality we present here. This generality is understood as the fact that several nonlinear plants and observers are covered by a single proof. Chapter 3 demonstrates and illustrates the generality of our results by showing that the imposed assumptions in here are satisfied for many plants and observers.

In this chapter, the estimation of the slow state of the plant is addressed by using an observer designed for the reduced order system and implemented on the original plant. Hence, we study the robustness of the observer with respect to singular perturbations since the fast variables are neglected during the observer design. We deal with a general class of plants with outputs disturbed by measurement noise so that we also study the robustness of the observer with respect to it. To the best of our knowledge, there are no existing results addressing this sort of robustness analysis within the singular perturbations framework.

To cover a wide class of systems, we consider singularly perturbed plants that have input-to-state practically stable (ISpS) slow systems and globally exponentially stable boundary layer systems. We assume that the error dynamics of the observer designed for the reduced system exhibit an input-to-state stability property (ISS) with linear gain from the measurement noise. We show that the error dynamics are ISpS stable when the observer is used on the original plant. Moreover, we exploit how our assumptions are stated to provide robustness results on finite-gain $\mathcal{L}_2$ stability.

Our main result implies that the error dynamics are practically globally exponentially stable when the measurement noise is equal to zero. Since our results cover a large class of plants and nonlinear observers, they constitute a new and general estimation framework for globally Lipschitz nonlinear singularly perturbed systems. Moreover, our results contribute to the existing literature that only applies to specific plants and observers. Although results in this chapter cover a smaller class of systems and observers than results presented later in Chapter 4, results in here are important contributions in their own right as they lead to strong conclusions for some classes of systems and observers that cannot be concluded with results presented in Chapter 4.

This chapter is organised as follows. Section 2.2 introduces the general class of nonlinear plants studied in here and the assumptions placed upon it. Section 2.3 shows a result on boundedness of solutions which is required to state the main result. Section 2.4 presents our main contribution on the observer robustness to singular perturbations and to measurement noise. In Section 2.5, we state the conclusions of the chapter.

2.2 General setting for globally Lipschitz nonlinear systems

Here, we deal with the estimation of the slow variables of singularly perturbed systems in the so-called standard form. Hence, consider the following general class of nonlinear singularly perturbed systems

$$\dot{x} = f_s(t, x, z, u, \varepsilon), \tag{2.1a}$$

$$\varepsilon\dot{z} = f_f(t, x, z, u, \varepsilon), \tag{2.1b}$$

$$y = h(t, x, z, u, w, \varepsilon), \tag{2.1c}$$

where $x \in \mathbb{R}^n$ represents the slow state of the plant, $z \in \mathbb{R}^m$ is the fast state, $y \in \mathbb{R}^p$ is the measured output, $u \in \mathbb{R}^r$ is the known input, $w \in \mathbb{R}^q$ is the measurement noise and $\varepsilon > 0$ is the singular perturbation parameter characterising the time scale separation. As far as we are aware, there are no existing results dealing with measurement noise in the context of nonlinear estimation of globally Lipschitz nonlinear singularly perturbed systems. The robustness analysis with respect to this sort of disturbances is a contribution of this work. We need to ensure that the measurement noise, the input and its derivative are bounded to perform our analysis and be able to conclude our results.

Assumption 2.1. *The input of the system (2.1),* $u \in \mathbb{R}^r$*, is differentiable and its derivative is bounded uniformly in* ε *for* $\varepsilon \ll 1$*. In addition, the input, its derivative and the measurement noise belong to* $\mathcal{L}_\infty$*; i.e.* $u, \dot{u}, w \in \mathcal{L}_\infty$.

Note that Assumption 2.1 is common and useful in singular perturbations theory when we intend to establish a result for the full system from assumptions over the reduced and boundary layer systems. Our goal is to analyse the performance of a nonlinear observer for the estimation of the slow variables of a two-time scale system when it has been designed for the reduced system. Hence, we investigate the robustness of the observer with respect to singular perturbations. Alongside with the singular perturbations analysis, we study the robustness of the observer to measurement noise ($w \in \mathbb{R}^q$). The conclusions in this chapter guarantee appropriate convergence properties of the estimation error when one picks any existing observer satisfying our assumptions to estimate the slow state $x \in \mathbb{R}^n$.

By following the singular perturbations technique, we decompose the plant (2.1) into the reduced (slow) order and the boundary layer systems which are associated with different time scales. We set $\varepsilon = 0$ to restrict the process dynamics to the slow manifold represented by the following algebraic equation

$$0 = f_f(t, x, z, u, 0). \tag{2.2}$$

Assumption 2.2. *The algebraic equation (2.2) has an isolated solution* $z = H(t, x, u)$ *that can be obtained analytically.*

Assumption 2.2 is common within the singular perturbation framework since it is needed to study the quasi-steady state performance of the system. Furthermore, we have assumed that $H(t, x, u)$ has a closed analytical representation. This is a needed strong assumption since we use a model-based observer. Note that this requirement can be relaxed to the case when the isolated root $H(t, x, u)$ is an approximated solution to (2.2). In that case, the robustness of the estimator to errors arising from such approximations opens an interesting topic for further research. We now substitute the isolated solution $z = H(t, x, u)$ in (2.1a) and (2.1c) at $\varepsilon = 0$. Hence, we obtain the reduced (slow) dynamical system given by

$$\dot{x} = f_s(t, x, H(t, x, u), u, 0), \tag{2.3a}$$

$$y_s = h(t, x, H(t, x, u), u, w, 0). \tag{2.3b}$$

We need to assume an appropriate stability property for the reduced system in order to be able to conclude appropriate results regarding the estimation error. Hence, we state the following assumption which is common within the observer design context.

Assumption 2.3. *For the reduced (slow) system (2.3), there exists a continuously differentiable function* $V_1(t, x)$, $b_i > 0$ *for* $i \in \{1, \dots, 5\}$, *and* $\delta_{V_1} \geq 0$ *such that for all* $x \in \mathbb{R}^n$, $u \in \mathbb{R}^n$, *and* $t \geq 0$

$$b_1|x|^2 \leq V_1(t, x) \leq b_2|x|^2, \tag{2.4}$$

$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x} f_s(t, x, H, u, 0) \leq -b_3|x|^2 + b_4|u|^2 + \delta_{V_1}, \tag{2.5}$$

$$\left| \frac{\partial V_1}{\partial x} \right| \leq b_5|x|. \tag{2.6}$$

Observe that any reduced system (2.3) satisfying Assumption 2.3 is globally input-to-state practically stable for any bounded input. The practical term in condition (2.5) implies that we can deal with plants that have reduced order systems that exhibit globally stable limit cycles. We require inequality (2.6) since we perform a robustness analysis for the full plant via Lyapunov methods. This condition is useful when using the Lipschitz properties of the plant to analyse the interconnection between the reduced system and the boundary layer system. Assumption 2.3 is standard within the estimation context since the observer design for nonlinear unbounded systems is a complicated problem to address.

To analyse the performance of the fast variables and their effect on the estimation error of the slow state, we consider the change of variables $\xi = z - H(t, x, u)$ so that the equilibrium of the fast dynamics is moved to the origin. Hence, the system (2.1) in the new coordinates (x, ξ) is written as follows

$$\dot{x} = f_s(t, x, \xi + H(t, x, u), u, \varepsilon), \tag{2.7a}$$

$$\begin{aligned} \varepsilon\dot{\xi} &= f_f(t, x, \xi + H(t, x, u), u, \varepsilon) - \varepsilon\frac{\partial H}{\partial t} - \varepsilon\frac{\partial H}{\partial u}\dot{u} \\ &\quad - \varepsilon\frac{\partial H}{\partial x} f_s(t, x, \xi + H(t, x, u), u, \varepsilon), \end{aligned} \tag{2.7b}$$

$$y = h(t, x, \xi + H(t, x, u), u, w, \varepsilon), \tag{2.7c}$$

where the quasi-steady-state of (2.7b) is $\xi = 0$, which when substituted into (2.7a) and (2.7c) leads to the reduced model (2.3). We need to ensure that $\varepsilon\dot{\xi}$ remain finite when $\varepsilon \to 0$ and $\dot{\xi} \to \infty$ to be able to analyse (2.7b). Hence, we study the fast dynamics performance by analysing the system (2.7) in the fast time scale $\tau := \frac{t - t_0}{\varepsilon}$. Observe that if $\varepsilon \to 0$, $\tau \to \infty$ even for finite t close to t_0. By considering the τ time scale, the singularly perturbed system (2.7a) - (2.7b) becomes

$$\frac{dx}{d\tau} = \varepsilon f_s(t, x, \xi + H(t, x, u), u, \varepsilon), \tag{2.8a}$$

$$\begin{aligned} \frac{d\xi}{d\tau} &= f_f(t, x, \xi + H(t, x, u), u, \varepsilon) - \varepsilon\frac{\partial H}{\partial t} - \varepsilon\frac{\partial H}{\partial u}\dot{u} \\ &\quad - \varepsilon\frac{\partial H}{\partial x} f_s(t, x, \xi + H(t, x, u), u, \varepsilon), \end{aligned} \tag{2.8b}$$

where we have used the approach in [33]. The variables t and x in (2.8b) are slowly time-varying since they are defined as follows

$$t = t_0 + \varepsilon\tau, \qquad x = x(t_0 + \varepsilon\tau).$$

Set $\varepsilon = 0$ so that the variables t and x are frozen to $t = t_0$ and $x = x(t_0)$. Therefore, (2.8a) becomes $dx/d\tau = 0$ and (2.8b) is reduced to the following autonomous system

$$\frac{d\xi}{d\tau} = f_f(t_0, x(t_0), \xi + H(t_0, x(t_0), u), u, 0), \tag{2.9}$$

which has an equilibrium point at $\xi = 0$. When $\xi = 0$ is asymptotically stable and $\xi(0)$ belong to its domain of attraction, the solutions to (2.9) converges to an $O(\varepsilon)$ neighbourhood of the origin during the boundary layer interval [70]. Beyond this interval, the slowly varying parameters (t, x) move away from their initial values (t_0, x_0). Then, we need to assume that (2.9) satisfy a stability property so that its solutions remain close to the origin even when (t, x) has changed. To analyse this situation, the frozen variables $t = t_0$ and $x = x(t_0)$ must be allowed to take values in the region of the slowly varying parameters (t, x). Hence, we rewrite (2.9) as follows

$$\frac{d\xi}{d\tau} = f_f(t, x(t), \xi + H(t, x(t), u), u, 0), \tag{2.10}$$

where (t, x) are treated as fixed parameters. The system (2.10) is the corresponding boundary layer model for the system (2.7). The reader can refer to [70, 75] for further details on how the boundary layer system is obtained. To justify the model reduction so that the performance of the plant (2.7) can be approximated by (2.3) and (2.10), a critical stability property needed for (2.10) is stated in the following assumption.

Assumption 2.4. *For the boundary layer system (2.10), there exists a Lyapunov function* $W(t, x, \xi)$ *and* $c_i > 0$ *for* $i \in \{1, \dots, 6\}$ *such that for all* $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^m$, $u \in \mathbb{R}^r$, *and* $t \geq 0$

$$c_1|\xi|^2 \leq W(t, x, \xi) \leq c_2|\xi|^2, \tag{2.11}$$

$$\frac{\partial W}{\partial \xi} f_f(t, x, \xi + H(t, x, u), u, 0) \leq -c_3|\xi|^2, \tag{2.12}$$

$$\left|\frac{\partial W}{\partial \xi}\right| \leq c_4|\xi|, \quad \left|\frac{\partial W}{\partial t}\right| \leq c_5|\xi|^2, \quad \left|\frac{\partial W}{\partial x}\right| \leq c_6|\xi|. \tag{2.13}$$

Remark 2.1. *Although Assumption 2.4 is strong due to the uniformity in* u *in (2.12), it is a common assumption in singular perturbations analysis [33]. If we claim that (2.10) is exponentially stable uniformly in* t, x *and* u, *we can use the converse Lyapunov theorem [Lemma 9.8, 70] in which is stated that* $\left|\frac{\partial W}{\partial x}\right| \leq c|\xi|^2$ *for some* $c > 0$. *This is a slightly more general condition than the last inequality in (2.13). However, we need condition (2.13) for our proof to state a global result.*

2.3 Practical DISS and practical $\mathcal{L}_2$ stability of the plant

Our main result requires the solutions of the system (2.7) to be bounded for all time $t \geq 0$. Hence, we need to guarantee a boundedness of solutions property to be able to conclude the main result of this chapter. Although one can directly assume the solutions of the plant are bounded, we have stated and shown this result for (2.7) since it is important in its own right and can be established from the same assumptions needed for our main result. We characterise the boundedness of solutions of the plant by showing practical DISS and practical $\mathcal{L}_2$ stability results.

Here, we perform a Lyapunov analysis so that we define a composite Lyapunov function by using the Lyapunov functions satisfying Assumptions 2.3 and 2.4. Then, we take the time derivative of the composite Lyapunov function along the trajectories of (2.7). We need to take into account the terms that arise from the interconnection of the slow and fast dynamics since our assumptions are made for the lower dimensional systems. In general, those terms are sign indefinite, so we need to appropriately bound them.

Assumption 2.5. *Consider* $f_s(t, x, \xi + H(t, x, u), u, \varepsilon)$ *and let* $L_i \geq 0$*, for* $i \in \{0, 1, 2\}$*, be such that the following inequalities hold for all* $x \in \mathbb{R}^n$*,* $\xi \in \mathbb{R}^m$*,* $u \in \mathbb{R}^r$*, and* $t \geq 0$

$$|f_s(t, x, \xi + H(t, x, u), u, \varepsilon) - f_s(t, x, \xi + H(t, x, u), u, 0)| \leq \varepsilon L_0(|x| + |\xi| + |u|), \tag{2.14}$$

$$|f_s(t, x, \xi + H(t, x, u), u, 0) - f_s(t, x, H(t, x, u), u, 0)| \leq L_1|\xi|, \tag{2.15}$$

$$|f_s(t, x, H(t, x, u), u, 0)| \leq L_2(|x| + |u|). \tag{2.16}$$

Assumption 2.6. *Consider* $f_f(t, x, \xi + H(t, x, u), u, \varepsilon)$ *and* $H(t, x, u)$*. Let* $L_i \geq 0$*, for* $i \in \{3, \ldots, 6\}$*, be such that the following inequalities hold for all* $x \in \mathbb{R}^n$*,* $\xi \in \mathbb{R}^m$*,* $u \in \mathbb{R}^r$*, and* $t \geq 0$

$$|f_f(t, x, \xi + H(t, x, u), u, \varepsilon) - f_f(t, x, \xi + H(t, x, u), u, 0)| \leq \varepsilon L_3(|x| + |\xi| + |u|), \tag{2.17}$$

$$\left|\frac{\partial H}{\partial t}\right| \leq L_4(|x| + |u|), \qquad \left|\frac{\partial H}{\partial x}\right| \leq L_5, \qquad \left|\frac{\partial H}{\partial u}\right| \leq L_6. \tag{2.18}$$

Remark 2.2. *The given inequalities in Assumptions 2.5 and 2.6 can be deduced from general conditions over the maps* $f_s(t, x, \xi + H(t, x, u), u, \varepsilon)$*,* $f_f(t, x, \xi + H(t, x, u), u, \varepsilon)$ *and* $H(t, x, u)$*. Observe that if* $f_s(t, x, \xi + H(t, x, u), u, \varepsilon)$ *and* $f_f(t, x, \xi + H(t, x, u), u, \varepsilon)$ *are continuously differentiable with globally bounded derivatives,* $f_s(0, 0, 0, 0, \varepsilon) = 0$*,* $f_f(0, 0, 0, 0, \varepsilon) = 0$ *and* $H(t, 0, 0) = 0$*, we can conclude (2.14) - (2.17). Similarly, conditions in (2.18) can be deduced if* $\partial H/\partial x$ *and* $\partial H/\partial u$ *have continuous and bounded first* *partial derivatives with respect to* t *[35].*

We are now ready to present our first result on boundedness of solutions of the system (2.7). This result is stated by using the previous assumptions for the lower dimensional systems (2.3) and (2.10). Note that we use the result in Lemma 2.1 to prove the main result of this chapter.

Lemma 2.1. *Consider the singularly perturbed system (2.7). If Assumptions 2.1 - 2.6 hold, there exists* $\tilde{\varepsilon}^* > 0$, $k_i > 0$ *for* $i \in \{1,2\}$, $k_i \geq 0$ *for* $i \in \{3,\ldots,7\}$, $\ell_1 > 0$, $\ell_i \geq 0$ *for* $i \in \{2,\ldots,6\}$, *such that the solutions to the system (2.7) satisfy the following ISS and* $\mathcal{L}_2$ *conditions*[1]

$$|(x(t),\xi(t))| \leq k_1 \exp\left[-k_2(t-t_0)\right] |(x_0,\xi_0)| + \left(k_3 + \varepsilon^{\frac{1}{2}} k_4 + \varepsilon k_5\right) |u[t_0,t]| \\ + \varepsilon^{\frac{1}{2}} k_6 |\dot{u}[t_0,t]| + k_7, \tag{2.19}$$

$$|(x(t),\xi(t))|_{\mathcal{L}_2} \leq \ell_1 |(x_0,\xi_0)| + \left(\ell_2 + \varepsilon^{\frac{1}{2}}\ell_3 + \varepsilon \ell_4\right) |u(t)|_{\mathcal{L}_2} + \varepsilon^{\frac{1}{2}} \ell_5 |\dot{u}(t)|_{\mathcal{L}_2} + \ell_6 t, \tag{2.20}$$

for all $\varepsilon \in (0,\tilde{\varepsilon}^*)$ *and for all* $(x_0,\xi_0) \in \mathbb{R}^n \times \mathbb{R}^m$, $u, \dot{u} \in \mathcal{L}_\infty$, *and* $t \geq 0$.

Note that (2.19) implies that the singularly perturbed system (2.7) is practical input-to-state stable with respect to the input and its derivative (practical DISS), while (2.20) means that (2.7) is practical $\mathcal{L}_2$ stable. The proof of Lemma 2.1 is presented in Appendix A.1. The following result on the trajectories of the fast dynamics is a direct implication of Lemma 2.1.

Corollary 2.1. *Consider the singularly perturbed system (2.7). If Assumptions 2.1 - 2.6 hold, there exists* $\bar{\varepsilon}^* > 0$, $\tilde{k}_i > 0$ *for* $i \in \{1,2\}$, $\tilde{k}_i \geq 0$ *for* $i \in \{3,4,5\}$, $\tilde{\ell}_1 > 0$ *and* $\tilde{\ell}_i \geq 0$ *for* $i \in \{2,\ldots,5\}$, *such that the fast state satisfies the following ISS and* $\mathcal{L}_2$ *conditions*

$$|\xi(t)| \leq \tilde{k}_1 \exp\left(-\tilde{k}_2 \frac{t-t_0}{\varepsilon}\right) |\xi_0| + \left(\varepsilon \tilde{k}_3 + \varepsilon^2 \tilde{k}_4\right) \left(|x[t_0,t]| + |u[t_0,t]|\right) \\ + \varepsilon \tilde{k}_5 |\dot{u}[t_0,t]|, \tag{2.21}$$

$$|\xi(t)|_{\mathcal{L}_2} \leq \varepsilon^{\frac{1}{2}} \tilde{\ell}_1 |\xi_0| + \varepsilon\left(\tilde{\ell}_2 + \varepsilon \tilde{\ell}_4\right) |x(t)|_{\mathcal{L}_2} + \varepsilon \tilde{\ell}_3 |\dot{u}(t)|_{\mathcal{L}_2} + \varepsilon\left(\tilde{\ell}_2 + \varepsilon \tilde{\ell}_5\right) |u(t)|_{\mathcal{L}_2}, \tag{2.22}$$

for all $\varepsilon \in (0,\bar{\varepsilon}^*)$, $\xi_0 \in \mathbb{R}^m$, $x, u, \dot{u} \in \mathcal{L}_\infty$, *and* $t \geq 0$.

The result in Corollary 2.1 is useful to study the convergence of the estimation error in the next section as we need to analyse the effect of the fast state on the observer.

[1] In the sequel, $x_0 := x(0)$. The same apply for the other states.

Note that conditions (2.21) and (2.22) imply that the fast state rapidly becomes upper bounded by an $O(\varepsilon)$ term. This property of the fast state is crucial for our main result since, for instance, we use the fact that the exponential function in (2.21) rapidly converges to zero to remove undesired terms bounding the estimation error. The proof of Corollary 2.1 is presented in Appendix A.2.

Remark 2.3. *We have taken advantage of our assumptions to prove practical DISS and practical $\mathcal{L}_2$ stability results for the singularly perturbed plant (2.7). Lemma 2.1 and Corollary 2.1 state practical DISS results for the full state and the fast variable, respectively. It is observed that if we set $\varepsilon = 0$ in (2.19), we recover the property implied by Assumption 2.3. The $\mathcal{L}_2$ results are equivalent to the ISS ones, but from the perspective of finite-gain $\mathcal{L}$ stability. Note that $\mathcal{L}_2$ bounds are very useful when dealing with $\mathcal{L}_2$ bounded noise and in optimisation problems. Both ISS and $\mathcal{L}_2$ conditions prove different robustness properties of the system which are used to conclude appropriate results in our main result of the chapter in Theorem 2.1.*

2.4 Estimation error convergence result

Here, we focus on the estimation of the slow state of the plant (2.7) by using a full-order nonlinear observer for globally Lipschitz systems synthesized based on (2.3). We now give a general set of assumptions that cover a large class of full-order observers described by

$$\dot{\hat{x}} = f_o(t, \hat{x}, y_s, u), \tag{2.23}$$

where $\hat{x} \in \mathbb{R}^n$ is the state of the observer and an estimate of $x \in \mathbb{R}^n$ (slow variable), y_s is the output of the reduced order system (2.3) and u is the input to the system. The nonlinear estimation problem studied here is summarised in Figure 2.1.

We define the estimation error as $e = x - \hat{x}$. Therefore, the error dynamics for the observer synthesised for the reduced system (2.3) is given by

$$\dot{e} = f_e(t, x, e, H(t, x, u), y_s, u, 0), \tag{2.24}$$

where $f_e = f_s(t, x, H(t, x, u), u, 0) - f_o(t, x - e, y_s, u)$. When we synthesise the error dynamics, we consider the reduced system (2.3) as if the slow state were independent of the fast part of the plant. Note that the last argument of f_e, which is set to zero, explicitly

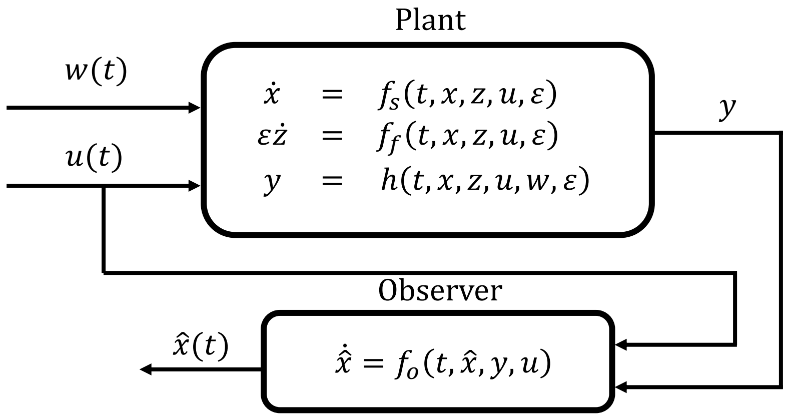

Figure 2.1: Block diagram of the estimation of the slow variables of a globally Lipschitz nonlinear singularly perturbed system.

indicates that the error dynamics (2.24) refer to the observer designed for the reduced system, i.e., when the perturbation parameter ε is equal to zero.

The interconnection of the reduced system (2.3) and the observer (2.23) can be represented trough the extended state $(x, e) \in \mathbb{R}^n \times \mathbb{R}^n$. It follows that (x, e) leads to the following extended system

$$\dot{x} = f_s(t, x, H(t, x, u), u, 0), \tag{2.25a}$$

$$\dot{e} = f_e(t, x, e, H(t, x, u), y_s, u, 0), \tag{2.25b}$$

$$y_s = h(t, x, H(t, x, u), u, w, 0). \tag{2.25c}$$

Note that the estimation error dynamics are in cascade with the reduced system as (2.25a) does not depend on e. This property is preserved when the observer (2.23) is implemented on the full plant (2.7). Then, we take advantage of it in the convergence analysis of the estimation error.

We now assume a set of appropriate properties for the error dynamics (2.24) to ensure a desired performance of the observer (2.23) when we use it to estimate the slow state of the singularly perturbed system (2.7). The following assumption characterises the stability of the observer for the reduced system (2.3).

Assumption 2.7. *For the error dynamics (2.24), there exists a continuously differentiable function* $V_e(t, e)$ *and* $a_i > 0$ *for* $i \in \{1, \ldots, 5\}$*, such that for all* $(x, e) \in \mathbb{R}^n \times \mathbb{R}^n$, $u \in \mathbb{R}^r$,

$w \in \mathbb{R}^q$*, and* $t \geq 0$

$$a_1|e|^2 \leq V_e(t,e) \leq a_2|e|^2, \tag{2.26}$$

$$\frac{\partial V_e}{\partial t} + \frac{\partial V_e}{\partial e} f_e(t,x,e,H,y_s,u,0) \leq -a_3|e|^2 + a_4|w|^2, \tag{2.27}$$

$$\left|\frac{\partial V_e}{\partial e}\right| \leq a_5|e|. \tag{2.28}$$

Remark 2.4. *Assumption 2.7 implies that the error dynamics (2.24) are ISS stable respect to the measurement noise with an exponential* $\mathcal{KL}$ *function and a linear gain from* $w \in \mathbb{R}^q$ *(measurement noise). Furthermore, as the gain from* $w \in \mathbb{R}^q$ *in (2.27) is quadratic, Assumption 2.7 implies an* $\mathcal{L}_2$ *stability property with linear gain from* $w \in \mathbb{R}^q$ *to* $e \in \mathbb{R}^n$*. Note that the measurement noise affect the error dynamics via the output of the system which is seen as an input to the observer dynamics (2.23). It is important to highlight that the estimation problem of linear/nonlinear singularly perturbed systems in the presence of measurement noise has not been considered before. Therefore, its inclusion is a contribution of this work.*

Since the observer (2.23) designed for the slow system (2.3) is then implemented on the full system (2.7), the singular perturbation parameter ε and the fast state ξ have an effect on the performance of the slow state estimation error. Then, it follows that the error dynamics for the observer when it is used on the original plant (2.7) is given by

$$\dot{e} = f_e(t,x,e,\xi + H(t,x,u),y,u,\varepsilon), \tag{2.29}$$

where $f_e = f_s(t,x,\xi + H(t,x,u),u,\varepsilon) - f_o(t,x-e,y,u)$. While (x,e) represents the interconnection between the reduced system (2.3) and the error dynamics (2.24), the extended state (x,e,ξ) characterises the interconnection between the singularly perturbed plant (2.7) and the error dynamics (2.29). Hence, the full extended interconnected system is given by

$$\dot{x} = f_s(t,x,\xi + H(t,x,u),u,\varepsilon), \tag{2.30a}$$

$$\dot{e} = f_e(t,x,e,\xi + H(t,x,u),y,u,\varepsilon), \tag{2.30b}$$

$$\begin{aligned}\varepsilon\dot{\xi} = {} & f_f(t,x,\xi + H(t,x,u),u,\varepsilon) - \varepsilon\frac{\partial H}{\partial t} - \varepsilon\frac{\partial H}{\partial u}\dot{u} \\ & - \varepsilon\frac{\partial H}{\partial x} f_s(t,x,\xi + H(t,x,u),u,\varepsilon),\end{aligned} \tag{2.30c}$$

$$y = h(t,x,\xi + H(t,x,u),u,w,\varepsilon). \tag{2.30d}$$

Since the quasi-steady-state of (2.30c) is $\xi = 0$, by setting $\varepsilon = 0$ in (2.30), we recover the interconnected system (2.25). We need to analyse the interconnected system (2.30) to conclude a result regarding the convergence properties of the slow state estimation error. Hence, we have to introduce an assumption to relate (2.30) with the error dynamics (2.24) and Assumption 2.7.

Assumption 2.8. *Let* $L_7 \geq 0$ *and* $L_8 \geq$ *be such that* $f_o(t, x - e, y, u)$ *satisfies the following inequality for all* $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^m$, $e \in \mathbb{R}^n$, $u \in \mathbb{R}^r$, $w \in \mathbb{R}^q$, *and* $t \geq 0$

$$|f_o(t, x - e, y, u) - f_o(t, x - e, y_s, u)| \leq \varepsilon L_7 + L_8|\xi|, \tag{2.31}$$

where $y = h(t, x, \xi + H(t, x, u), u, w, \varepsilon)$ *and* $y_s = h(t, x, H(t, x, u), u, w, 0)$.

Remark 2.5. *Inequality (2.31) can be obtained if the output map* $h(t, x, z, u, w, \varepsilon)$ *is globally Lipschitz in* $(x, z, u, w, \varepsilon)$ *uniformly in* t *and if* $f_o(t, \hat{x}, y, u)$ *is globally Lipschitz in* $(\hat{x}, y, u)$ *uniformly in* t.

We now use the assumptions and results in the previous section as well as the assumptions in this section to state the main result of this chapter. In Theorem 2.1, we conclude that the estimation error dynamics exhibit a practical DISS and a practical $\mathcal{L}_2$ stability properties when the observer (2.23) is used to estimate the slow states of the singularly perturbed plant (2.7). These results imply interesting properties of the estimation error when the output is not corrupted by measurement noise. For instance, the practical DISS property leads to practical asymptomatic stability of the error dynamics. Further discussions on this matter are presented in Remark 2.6. The proof of Theorem 2.1 is presented in Appendix A.3.

Theorem 2.1. *Consider the singularly perturbed system (2.30). If Assumptions 2.1 - 2.8 hold, there exists* $\varepsilon^* > 0$, $\bar{k}_i > 0$ *for* $i \in \{1, 2\}$, $\bar{k}_i \geq 0$ *for* $i \in \{3, \ldots, 6\}$, $\hat{\ell}_1 > 0$ *and* $\hat{\ell}_i \geq 0$ *for* $i \in \{2, \ldots, 5\}$, *such that the error dynamics satisfy the following ISS and* $\mathcal{L}_2$ *stability properties*

$$\begin{aligned}|e(t)| \leq{}& \bar{k}_1 \exp\left[-\bar{k}_2(t - t_0)\right] |e_0| + \varepsilon\bar{k}_3 + \varepsilon\bar{k}_4 \Big[|x[t_0, t]| + |\xi[t_0, t]| + |u[t_0, t]|\Big] \\ &+ \bar{k}_5|\xi[t_0, t]| + \bar{k}_6|w[t_0, t]|, \end{aligned} \tag{2.32}$$

$$\begin{aligned}|e(t)|_{\mathcal{L}_2} \leq{}& \hat{\ell}_1|e_0| + \hat{\ell}_2|w(t)|_{\mathcal{L}_2} + \varepsilon\hat{\ell}_3 \Big[|x(t)|_{\mathcal{L}_2} + |\xi(t)|_{\mathcal{L}_2} + |u(t)|_{\mathcal{L}_2}\Big] \\ &+ \hat{\ell}_5|\xi(t)|_{\mathcal{L}_2} + \varepsilon\hat{\ell}_4 t, \end{aligned} \tag{2.33}$$

for all $\varepsilon \in (0, \varepsilon^*)$, $e_0 \in \mathbb{R}^n$, $x, \xi, u, \dot{u} \in \mathcal{L}_\infty$ *and* $t \geq 0$.

Remark 2.6. *The error dynamics are in cascade with the state* (x, ξ). *Hence,* $x(t)$ *and* $\xi(t)$ *are seen as signals in Theorem 2.1. It follows from Lemma 2.1 that* $x(t)$ *and* $\xi(t)$ *are bounded signals when considered as inputs to the error dynamics for any initial conditions and bounded inputs with bounded derivatives. By (2.21), the fast states rapidly converge to a ball of order* ε *since the exponential function in (2.21) quickly converges to zero in a finite time* $\varepsilon T_\xi > 0$ *where* T_ξ *depends on* ξ_0. *Note that the convergence rate can be adjusted by reducing* ε. *Then, it can be proven that the* $O(1)$ *term in* ε *that depends on* $\xi(t)$ *in (2.32) rapidly converges. It is observed in (2.32) that the gains from* x *to* e *are of* $O(\varepsilon)$. *Then, the effect of* x *over the error dynamics can be attenuated by reducing* ε. *Moreover, Lemma 2.1 implies there is a finite time* $T > 0$ *depending on* (x_0, ξ_0) *such that the contribution due to* x *becomes of* $O(\varepsilon)$ *for all* $t \geq T$. *Hence, for all* $t > T$ *the ultimate bound for the estimation error is an* $O(\varepsilon)$ *term plus the input gain due to the measurement noise. Note that the* $O(\varepsilon)$ *in the ultimate bound only depends on* $\bar{k}_3$, u *and* $\dot{u}$. *By using an ISS approach for interconnected systems as in [Lemma 4.7, 70], it can be shown that (2.32) can be written in a form such that if* ε *is arbitrarily small, the property implied by Assumption 2.7 is recovered.*

Remark 2.7. *Theorem 2.1 implies "practical"* $\mathcal{L}_2$ *stability, where the practical meaning is understood based on the definition [Property* I_3*, 95]. Note that the practical term in (2.33) is of* $O(\varepsilon)$. *By using the cascade properties of the error dynamics, it can be proven that (2.33) has an equivalent form in which the input gains from* u, $\dot{u}$ *and* (x_0, ξ_0) *to* e *are of* $O(\varepsilon)$. *Moreover, Lemma 2.1 implies the gain from* ξ_0 *to* e *is of* $O(\varepsilon^{\frac{1}{2}})$. *It follows that we can find a finite time* $T > 0$ *such that the* $\mathcal{L}_2$ *bound becomes a term of* $O(\varepsilon)$ *plus the contribution due to the measurement noise. Since the contribution from the disturbance to the error dynamics has a finite gain in (2.33), it implies that the observer is robust with respect to measurement noise when the described approach is used.*

Remark 2.8. *In the absence of the measurement noise, Theorem 2.1 implies global exponential practical stability of the error dynamics.*

2.5 Conclusions of the chapter

We considered the estimation of the slow state of nonlinear singularly perturbed plants with appropriate global Lipschitz properties. The estimation problem was addressed by using a nonlinear observer synthesised for the reduced system and implemented on the full plant. We dealt with a general class of full-order observers for globally Lipschitz nonlinear systems and analysed their robustness with respect to singular perturbations. Moreover, we studied the robustness of the observer to measurement noise since it was assumed that the output of the plant is corrupted by this sort of disturbances. As far as we are aware, this is the first time that measurement noise is considered in the singular perturbation framework for nonlinear observer design.

Under a set of global assumptions, we proved boundedness of solutions results in Lemma 2.1 and Corollary 2.1 as well as the convergence properties of the estimation error in Theorem 2.1. We showed that the error dynamics exhibit practical DISS and practical $\mathcal{L}_2$ stability properties in Theorem 2.1. These results imply that the estimation error globally practically converges in the absence of measurement noise. Although the observer design process does not consider the singular perturbation parameter and the fast variables, we proved that the observer performs well when implemented on the full system. In conclusion, we have generated a general and solid design framework that allows us to estimate the slow states of globally Lipschitz nonlinear singularly perturbed systems by using a number of existing nonlinear observers in the literature.

Chapter 3

Applications of Global Results

In this chapter, we focus on presenting different classes of plants and full-order nonlinear observers that are covered by the results in Chapter 2. We give the conditions so that the stated assumptions in Chapter 2 are verified. Moreover, we include simulation results to illustrate our theoretical findings.

3.1 Introduction

THE THEORETICAL results developed in Chapter 2 lead to a design framework for the observer design of full-order observers for nonlinear Lipschitz singularly perturbed systems. We have presented a design framework and we have given conditions to cover a large class of plants and observers. Here, we demonstrate and support the generality and usefulness of our theoretical results by showing that our Assumptions in Chapter 2 hold for several situations. We show that our results apply to nonlinear singularly perturbed systems with reduced order models which have the appropriate structure so that many existing nonlinear observer design methods can be used within our estimation framework.

We study four classes of nonlinear singularly perturbed systems with reduced order models for which we can design the following nonlinear observers

- High-gain observer for Lipschitz nonlinear systems [2].
- Circle-criterion observer for systems with global Lipschitz nondecreasing nonlinearities [9].
- Circle criterion-based $\mathcal{H}_\infty$ observer for Lipschitz nonlinear systems with linear output [128].
- Circle criterion-based $\mathcal{H}_\infty$ observer for Lipschitz nonlinear systems with nonlinear output [128].

Each section of this chapter contains the study of the aforementioned nonlinear observers and their corresponding simulation results. We have reported in a conference paper that our results from Chapter 2 cover the class of plants and the nonlinear observer considered in [68] when we assume global Lipschitz properties for the reduced order model. This is not presented in here as we study its local version in Chapter 5.

3.2 High-gain observer for Lipschitz nonlinear systems

We first analyse the nonliner observer introduced in [2] and the class of nonlinear singularly perturbed plants in the following form

$$\dot{x} = Ax + f(t, x, z), \tag{3.1a}$$

$$\varepsilon\dot{z} = M_1 x + M_2 z, \tag{3.1b}$$

$$y = C_1 x + C_2 z + Dw, \tag{3.1c}$$

where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$ are the slow and fast state vectors, $y \in \mathbb{R}^p$ is the measured output, $w \in \mathbb{R}^p$ is the measurement noise which is assumed to be globally bounded, $0 < \varepsilon \ll 1$ is the singular perturbation parameter of the plant, and A, C_1, C_2, D, M_1 and M_2 are matrices of appropriate dimensions. Observe that the plant (3.1) is covered by the general class of systems (2.1).

Assumption 3.1. *The map* $f(t, x, z)$ *is continuous, globally Lipschitz and* $f(t, 0, 0) = 0$.

Assumption 3.2. *The matrix* M_2 *is Hurwitz.*

We have considered a linear fast dynamics in (3.1b) for two reasons: 1) it is easier to compute the slow manifold, and 2) after the model reduction the slow system has the structure for which we can design the high-gain observer introduced in [2]. Assumption 3.2 is required to approximate the system via the lower dimensional systems since the inverse of M_2 must exist.

Next, we demonstrate that Assumptions 2.1 - 2.6 are satisfied by the class of systems (3.1) so that we can use conclusions from Lemma 2.1 for this class of plants in (3.1). Then, we introduce the observer presented in [2] and show that Assumptions 2.7 and 2.8 hold. Therefore, we can use results in Theorem 2.1 to predict the performance of the estimation error. Note that there are no inputs to the system, then Assumption 2.1 trivially holds. We now set $\varepsilon = 0$ in (3.1b), such that the system is restricted to the slow manifold defined by

$$M_1x + M_2z = 0. \tag{3.2}$$

It follows that the algebraic equation (3.2) has an analytical solution given by $H(x) = -M_2^{-1}M_1x$. Hence, Assumption 2.2 holds. By using $H(x)$, we obtain the slow model defined as follows

$$\dot{x} = Ax + \bar{f}(t,x), \tag{3.3a}$$

$$y_s = Cx + Dw, \tag{3.3b}$$

with $\bar{f}(t,x) = f(t,x,-M_2^{-1}M_1x)$, and $C = C_1 - C_2M_2^{-1}M_1x$.

Assumption 3.3. *The function* $\bar{f}(t,x) = f(t,x,H(x))$ *in (3.3a) satisfies*

$$\bar{f}(t,x) = \begin{bmatrix} \bar{f}_1(t,x_1) \\ \bar{f}_2(t,x_1,x_2) \\ \vdots \\ \bar{f}_n(t,x_1,\dots,x_n) \end{bmatrix}. \tag{3.4}$$

Moreover, $f(t,x,H(x))$ *satisfies [Assumption 1, 2].*

Assumption 3.4. *The reduced system (3.3) is globally exponentially practically stable.*

Note that we need Assumption 3.4 to be able to design the observer for the reduced system (3.3). It follows from Assumption 3.4 that Assumption 2.3 holds. Now, define the change of variables $z = \xi - M_2^{-1}M_1x$. Therefore, we have

$$\dot{x} = Ax + f(t,x,\xi - M_2^{-1}M_1x), \tag{3.5a}$$

$$\varepsilon\dot{\xi} = M_2\xi + \varepsilon(M_2^{-1}M_1)[Ax + f(t,x,\xi - M_2^{-1}M_1x)], \tag{3.5b}$$

$$y = Cx + C_2\xi + Dw \tag{3.5c}$$

By using the fast time scale $\tau = t/\varepsilon$, it follows that the boundary layer system is given by

$$d\xi/d\tau = M_2\xi. \tag{3.6}$$

Since M_2 is Hurwitz, we have from [Theorem 4.6, 70] that for any given positive definite symmetric matrix Q_ξ there exists a positive definite symmetric matrix P_ξ that satisfies the Lyapunov equation:

$$P_\xi M_2 + M_2^T P_\xi = -Q_\xi. \tag{3.7}$$

Then, consider $W(\xi) = \xi^T P_\xi \xi$ as a candidate Lyapunov function for boundary layer system (3.6). It follows that

$$\frac{\partial W}{\partial \xi} M_2 \xi \le -\lambda_{\min}\{Q_\xi\}|\xi|^2, \tag{3.8}$$

so that Assumption 2.4 holds with $c_1 = \lambda_{\min}\{P_\xi\}$, $c_2 = \lambda_{\max}\{P_\xi\}$, $c_3 = -\lambda_{\min}\{Q_\xi\}$, $c_4 = 2\lambda_{\max}\{P_\xi\}$, $c_5 = 0$ and $c_6 = 0$. We now check inequalities in Assumptions 2.5 and 2.6. Note that $L_0 = 0$ and $L_3 = 0$ because the right-hand side of the system does not depend on ε. Since [Assumption 1, 2] holds, there is $L_{hg} > 0$ satisfying such an assumption so that (2.15) holds with $L_1 = L_{hg_1}$. Since $f(t, x, z)$ vanishes at $(x, z) = (0, 0)$, it follows that $|f(t, x, H(x))| \le L_{hg_2}$ so that (2.16) holds with $L_2 = L_{hg_2}$. The isolated solution $H(x)$ is a function of x; hence, $L_4 = 0$, $L_5 = |M_2^{-1} M_1|$ and $L_6 = 0$. Therefore, Assumptions 2.5 and 2.6 hold.

3.2.1 Observer design

We now focus on designing an observer for the reduced order system (3.3) and on studying its performance when implemented in the full system (3.1). So, consider the high gain full-order observer presented in [2] which has the following form

$$\dot{\hat{x}} = A\hat{x} + \bar{f}(t, \hat{x}) + G(\gamma, K)(y - C\hat{x}), \tag{3.9}$$

where $\hat{x} \in \mathbb{R}^n$ is the state of the observer and an estimate of the $x \in \mathbb{R}^n$, $K = [k_1, k_2, \dots, k_n]^T$ with $k_i \in \mathbb{R}$, for $i \in \{1, \dots, n\}$, to be suitably chosen, and

$$G(\gamma, K) := \begin{bmatrix} \gamma_1 k_1 \\ \gamma_2 k_2 \\ \vdots \\ \gamma_n k_n \end{bmatrix},$$

where γ_i is the i-th component of the design vector $\gamma \in \mathbb{R}^n$. Define the estimation error $\hat{e} = x - \hat{x}$, but similarly to the analysis in [2], the stability of the estimation error is studied through the transformed coordinates $T(\gamma)e = \hat{e}$ where $T(\gamma) = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_n)$. It is well know that a linear transformation preserves the stability property of the system (see [48]). The error dynamics in the transformed coordinates if defined as follows

$$\begin{aligned} \dot{e} &= \gamma_1(A - KC + \Omega(\gamma))e(t) + T(\gamma)^{-1}\left(\bar{f}(t, x(t) - f(t, x(t) - T(\gamma)e(t)\right) \\ &\quad + T(\gamma)^{-1}G(\gamma, K)Dw, \end{aligned} \tag{3.10}$$

where $\Omega(\gamma)$ is a matrix given by

$$\Omega := \begin{bmatrix} 0 & \alpha_1 & 0 & \dots & 0 \\ 0 & 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha_{n-1} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{3.11}$$

where $\alpha_i = \gamma_{i+1}/(\gamma_1\gamma_i) - 1$, for $i \in \{1, \dots, n-1\}$. From the definitions of $G(\gamma, K)$ and $T(\gamma)$, it is straightforward to conclude that $T(\gamma)^{-1}G(\gamma, K)Dw = KDw$. Therefore, (3.10) becomes

$$\dot{e} = \gamma_1(A - KC + \Omega(\gamma))e(t) + T(\gamma)^{-1}\left(\bar{f}(t, x(t) - f(t, x(t) - T(\gamma)e(t)\right) + KDw, \tag{3.12}$$

To design the high-gain observer (3.9), we consider the system (3.3) in the absence of measurement noise. From [Theorem 2, 2], we know that if there exists a constant $\lambda > 0$ and matrices $P = P^T > 0$, Y, $S = S^T > 0$ diagonal, and $W \leq 0$ diagonal such that

$$\begin{bmatrix} A^TP + PA + -C^TY^T - YC + \lambda I & \star \\ A_1^TP + WA_2 & -2S \end{bmatrix} < 0, \tag{3.13}$$

$$W > -S \tag{3.14}$$

and γ is chosen such that

$$\gamma_1 \geq \frac{1}{\lambda_{\min}(S^{-1}W) + 1}, \qquad \gamma_1 > \frac{2k_f\lambda_{\max}(P)}{\lambda},$$

with $k_f = \sqrt{\frac{n(n+1)}{2}} \max_{i\in\{1,\dots,n\}} L_i$ where L_i are the Lipschitz constants generated by [Assumption 1, 2]. By solving the LMIs in (3.13) and (3.14), we can choose the observer gains as $K = P^{-1}Y$ and $\gamma_i = \gamma_1^i \prod_{k=1}^{i-1}(\alpha_k + 1)$, for $i \in \{2, \dots, n\}$.

Remark 3.1. $\lambda > 0$*, which is computed via the LMI (3.13), is a small constant that determines the fast response of the High-gain observer (3.9). As* $\lambda > 0$ *is generated by using the reduced order system (3.3), we first fix the observer gain and then implement it in the original plant. Hence, there is no interaction between the singular perturbation parameter,* $\varepsilon > 0$*, and* $\lambda > 0$.

Note that, at this point, the observer has been designed by considering there is not measurement noise disturbing the output of (3.3). Hence, as the observer must be implemented on the system disturbed by the measurement noise, let consider $V(e) = e^T P e$ as a Lyapunov function for (3.12). Then, if the stated above holds, we conclude that the time derivative of $V(\cdot)$ along the trajectories of (3.12) is bounded as follows

$$\frac{\partial V_2}{\partial e} f_e(e) \leq -\nu |e|^2 + 2|e||w||KD|\lambda_{\max}\{P\}, \tag{3.15}$$

where $\nu > \gamma_1 - 2k_f \lambda_{\max}\{P\}/\lambda > 0$ and $f_e(e)$ is given by the right-hand side of (3.12). We use completion of squares on (3.15) so that we conclude that Assumption 2.7 holds with $\alpha_1 = \lambda_{\min}\{P\}$, $\alpha_2 = \lambda_{\max}\{P\}$, $\alpha_3 = \nu$, $\alpha_4 = 2\frac{|KD|^2 \lambda_{\max}\{P\}^2}{\nu}$ and $\alpha_4 = 2\lambda_{\max}\{P\}$. Since the output of the system does not depend on the perturbation parameter ε, it follows that $L_7 = 0$, and moreover, it is straightforward to see that $L_8 = |G(\gamma, K)||C_2|$. Therefore, Assumption 2.8 holds and our results apply and hold for the class of plants in (3.1) and the observer defined by (3.9). Notice that L_8 depends on γ, then a larger γ would lead to more conservative results and a smaller ε^* would be required for our results to hold.

3.2.2 Simulation results: A class of mechanical systems

We now illustrate the applicability of our results by performing simulations of the observer (3.9) when designed for an example that agrees with (3.1). Let us consider a mechanical system with a fast sensor dynamics described by the following model

$$\dot{x}_1 = x_2, \tag{3.16a}$$

$$\dot{x}_2 = x_3 - 0.1\sin(x_1 + z) - 0.2x_2, \tag{3.16b}$$

$$\dot{x}_3 = -0.1x_2 - 0.15x_3, \tag{3.16c}$$

$$\varepsilon \dot{z} = x_1 - z, \tag{3.16d}$$

$$y = z, \tag{3.16e}$$

When setting $\varepsilon = 0$, we obtain from (3.16d) that $H(x) = x_1$ is an isolated solution to $x_1 - z$. Hence, the reduced order system is given by

$$\dot{x}_1 = x_2, \tag{3.17a}$$

$$\dot{x}_2 = x_3 - 0.1\sin(2x_1) - 0.2x_2, \tag{3.17b}$$

$$\dot{x}_3 = -0.1x_2 - 0.15x_3, \tag{3.17c}$$

$$y = x_1, \tag{3.17d}$$

By following the framework described in Chapter 2, the high-order observer (3.9) must be designed for the reduced system (3.17) and implemented on the full system (3.16). Hence, we rewrite the system in the appropriate form that matches with (3.3). Then we have that

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad f(t,x) = \begin{pmatrix} 0 \\ -0.1\sin(2x_1) - 0.2x_2 \\ -0.1x_2 - 0.15x_3 \end{pmatrix}.$$

We apply the observer design method described above so that we obtain that the gains of the observer are given by $\gamma = [11.71, 107.8, 854.5]^T$ and $K = [2.88, 5.13, 1.43]^T$. We show simulation results in Figure 3.1, in which we have chosen different initial conditions to illustrate that the estimation error converges for any initial condition. Note that we have included measurement noise in our simulations such that the output of the system is: $y = z + 0.1\sin(10t) + 0.05\cos(5t)$. From our results, we found that the upper bound for ε is $\varepsilon^* = 0.059$. We performed our simulations at $\varepsilon = 0.01$ since this singular perturbation parameter is smaller than the given upper bound. Although we found that the observer for the reduced system converges immediately when implemented on (3.3), it is clear that the presence of the fast variable affects the performance of the estimation error when implemented on the full system. The initial conditions for the simulations presented in Figure 3.1 are given in Table 3.1.

	x_0	z_0	$\hat{x}_0$
Blue (a)	$(10, 5, 7)$	50	$(8, 14, 10)$
Red (b)	$(1, 10, 15)$	20	$(3, 7, 18)$
Orange (c)	$(30, 50, 21)$	80	$(30, 10, 3)$

Table 3.1: Initial conditions for the simulation results presented in Figure 3.1.

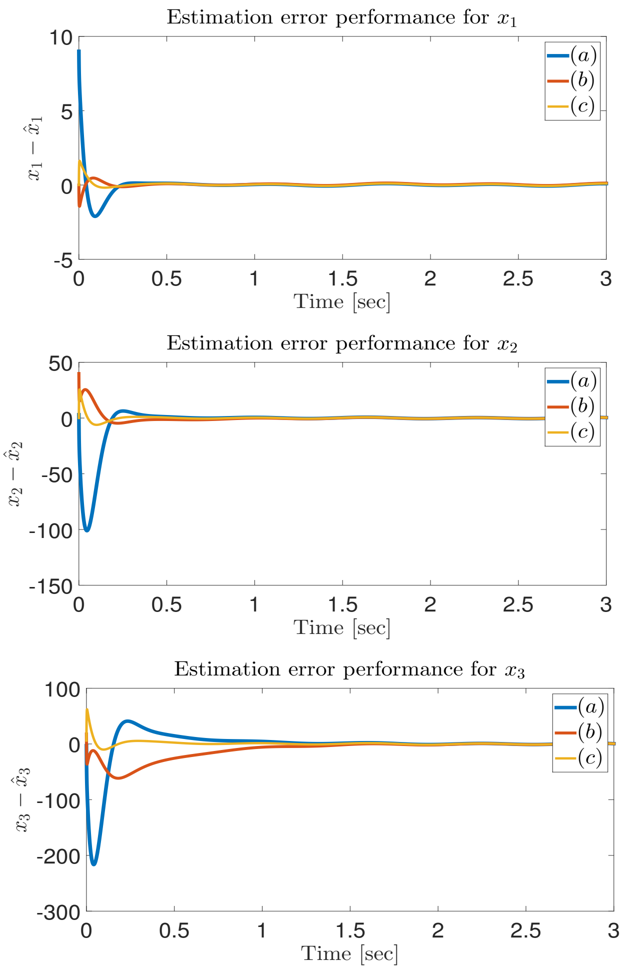

Figure 3.1: Simulations results for a mechanical system with a sensor with linear fast dynamics.

3.3 Circle-criterion observer for systems with global Lipschitz properties

In this section, we consider the class of plants with a singularly perturbed structure such that the slow model has a form in which results from [9] can be applied to design a full-order observer. We consider a class of plants covered by the general system (2.1), such class of plants is defined by

$$\dot{x} = Ax + G\gamma(Fx) + \sigma(y, u) + Bz, \tag{3.18a}$$

$$\varepsilon\dot{z} = M_1x + M_2z, \tag{3.18b}$$

$$y = C_1x + C_2z, \tag{3.18c}$$

where $x \in \mathbb{R}^n$ is the slow state, $z \in \mathbb{R}^m$ is the fast state, $y \in \mathbb{R}^p$ is the measured output variable, $u \in \mathbb{R}^r$ is the measured control input, $\varepsilon > 0$ is the singular perturbation parameter, $\gamma(\cdot)$ is a nondecreasing function, and A, B, C_1, C_2, F, G, M_1 and M_2 are matrices of appropriate dimensions. By having a linear fast dynamics, we can easily compute the slow manifold and end up with a reduced model with the appropriate structure for which we can design the circle-criterion observer in [9].

Assumption 3.5. *The functions $\gamma(\cdot)$ and $\sigma(\cdot,\cdot)$ are globally Lipschitz and vanish at zero. Moreover, the input belongs to $\mathcal{L}_\infty$; i.e. $u \in \mathcal{L}_\infty$.*

Assumption 3.6. *The matrix M_2 in (3.18b) is Hurwitz.*

Assumption 3.5 over $\sigma(\cdot,\cdot)$ is useful to prevent the solutions of $x \in \mathbb{R}^n$ from escaping to infinity in a finite time [9]. Moreover, Assumption 3.5 implies that $u \in \mathcal{L}_\infty$ so that Assumption 2.1 trivially holds since no condition is needed for $\dot{u}$ because the fast dynamics do no depend on u. Note that Assumption 3.6 is essential since the approximation of the singularly perturbed system (3.18) through lower dimensional systems is only possible if M_2 is Hurwitz.

To construct the reduced system and the boundary layer model, we first set $\varepsilon = 0$ in (3.18b) such that we obtain that the solutions to the system (3.18) are restricted to the slow manifold given by (3.2). It follows that $H(x) = -M_2^{-1}M_1x$ exists by virtue of Assumption 3.6. Subsequently, Assumption 2.2 holds. Then, the reduced order (slow) system is given by

$$\dot{x} = A_0x + G\gamma(Fx) + \sigma(y_s, u), \tag{3.19a}$$

$$y_s = Cx, \tag{3.19b}$$

where $A_0 = A - BM_2^{-1}M_1$ and $C = C_1 - C_2M_2^{-1}M_1$, and it assumed that the pair (A, C) is detectable.

Assumption 3.7. *The reduced system (3.19) is input-to-state practically stable (ISpS).*

By virtue of Assumption 3.7, there is a Lyapunov function such that the reduced system satisfies Assumption 2.3. This assumption allows more generality to the plant since there is no need for A_0 to be Hurwitz. We now define the change of variables

$z = \xi - M_2^{-1}M_1x$. Therefore, we have that the original system (3.18) in the (x, ξ) variables is given by

$$\dot{x} = Ax + G\gamma(Fx) + \sigma(y,u) + B\left(\xi - M_2^{-1}M_1x\right), \tag{3.20a}$$

$$\varepsilon\dot{\xi} = M_2\xi + \varepsilon\left(M_2^{-1}M_1\right)\left[Ax + G\gamma(Fx) + \sigma(y,u) + B\left(\xi - M_2^{-1}M_1x\right)\right], \tag{3.20b}$$

$$y = Cx + C_2\xi \tag{3.20c}$$

Following the same procedure as in the previous section, we consider the fast time scale $\tau = t/\varepsilon$ to obtain that the boundary layer system at $\varepsilon = 0$ is given again by (3.6). It is straightforward to check that Assumption 2.4 holds with with $c_1 = \lambda_{\min}\{P_\xi\}$, $c_2 = \lambda_{\max}\{P_\xi\}$, $c_3 = -\lambda_{\min}\{Q_\xi\}$, $c_4 = 2\lambda_{\max}\{P_\xi\}$, $c_5 = 0$ and $c_6 = 0$, where the matrices P_ξ and Q_ξ are defined by (3.7).

We now check the interconnection conditions in Assumptions 2.5 and 2.6. The right-hand side of the system does not depend on ε, then we have that $L_0 = 0$, and $L_3 = 0$. By Assumption 3.5, there is a Lipschitz constant $L_{c_1} > 0$ such that $G\gamma(Fx)| \leq L_{c_1}|x|$. Moreover, we have that $|\sigma(y,u) - \sigma(y_s,u)| \leq |C_2||\xi|$ and that there is $L_{c_2} > 0$ such that $|\sigma(y_s,u)| \leq \max\{|C|, L_{c_2}\}(|x| + |u|)$. Then, $L_1 = |C_2| + |B|$ where we have included $|B|$ since $|B\xi| \leq |B||\xi|$, and $L_2 = \max\{L_{c_1}, L_{c_2}\}$. Since the isolated solution $H(x)$ is a function of x we have $L_4 = 0$, $L_5 = \left|M_2^{-1}M_1\right|$ and $L_6 = 0$. Therefore, it follows that Assumptions 2.5 and 2.6 hold.

3.3.1 Observer design

Since the lower dimensional systems that approximate the plant (3.18) satisfy the given conditions in Chapter 2, we are able to design an observer to estimate the slow vriables of the system based on the reduced order model (3.19) . Then, consider the circle criterion observer introduced in [9] with the following dynamics

$$\dot{\hat{x}} = A_0\hat{x} + L(C\hat{x} - y_s) + G\gamma(F\hat{x} + K(C\hat{x} - y_s)) + \sigma(y_s,u), \tag{3.21}$$

where $\hat{x} \in \mathbb{R}^n$ is the observer's state and an estimate of $x \in \mathbb{R}^n$, K and L are gain matrices of appropriate dimensions which must be designed as described in the following. We need to design the observer (3.21) for the reduced system (3.19), and then implement it on the full singularly perturbed system (3.18). Define the estimation error as $e := x - \hat{x}$, such that the error dynamics are defined as follows

$$\dot{e} = (A_0 + LC)e + G\left[\gamma(Fx) - \gamma(F(x - e) - KCe)\right]. \tag{3.22}$$

To verify Assumption 2.7, we consider the Lyapunov function $V_2(e) = e^T P_3 e$, where the matrix $P_3 = P_3^T > 0$ is computed by solving the following LMI given in [9]

$$\begin{bmatrix} (A_0 + LC)^T P_3 + P_3(A_0 + LC) + \hat{\nu} & P_3 G + (F + KC)^T \Lambda \\ G^T P_3 + \Lambda(F + KC) & 0 \end{bmatrix} \leq 0, \tag{3.23}$$

where $\Lambda > 0$ is an observer design diagonal matrix, and $\hat{\nu} > 0$ is an observer design parameter. It follows from [9] that, when the observer is designed via the LMI (3.23), we have

$$\frac{\partial V_2}{\partial e} f_e(x, e) \leq -\hat{\nu}|e|^2, \tag{3.24}$$

with $f_e(x, e) = (A_0 + LC)e + G[\gamma(Fx) - \gamma(F(x - e) - KCe)]$. Hence, Assumption 2.7 holds with $a_1 = \lambda_{\min}\{P_3\}$, $a_2 = \lambda_{\max}\{P_3\}$, $a_3 = \hat{\nu}$, $a_4 = 0$, and $a_5 = 2|P_3|$. We now have to verify Assumption 2.8. Since the output of the system does not depend on ε, $L_7 = 0$. On the other hand, we have that $L_8 = |LC_2| + |G||KC_2| + |C_2|$ such that (2.31) is satisfied; then, Assumption 2.8 holds. Therefore, all the assumptions hold and our main result does too.

3.3.2 Simulation results

In this section, we present a seat suspension system in which we consider the ageing effects on a damper. The ageing process evolves in a much more slow time-scale than the operation of the system. Even though we consider a simple model of the seat suspension system, this application is significant because several problems that consider the ageing effects exhibit a similar singularly perturbed structure.

A simplified seat suspension model is shown in Figure 3.2. The system is composed of a single degree of freedom body mass, a linear spring and two dampers. Such a model has been extensively discussed in the literature and captures many essential characteristics of a real seat suspension system. To guarantee that the suspension system always have a dissipative element, we assume that one of the dampers of the seat suspension system suffers ageing effects while the other one can conserve its damping properties

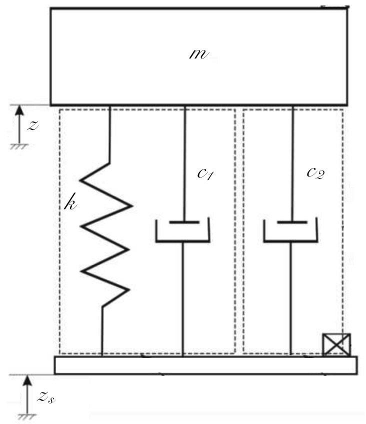

Figure 3.2: Simplified model of a semi-active seat suspension system.

over time. Otherwise, after a finite time, the system would become a simple oscillator. This assumption is understandable as real suspension systems always have back up dampers and springs.

We select the state variables as $z_1 := z - z_s$ and $z_2 := \dot{z}$, and define the disturbance caused by road roughness as $\hat{u} = z_s$ and $u = \dot{z}_s$. It is observed that the state space equation of the seat suspension can be written in the following form,

$$\dot{z}(t) = Az(t) + B \begin{bmatrix} \hat{u} \\ u \end{bmatrix} \tag{3.25}$$

with

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c_1+c_2(t)}{m} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & -1 \\ 0 & \frac{c_1+c_2(t)}{m} \end{bmatrix}.$$

where k is the constant of the spring, c_1 (constant) and $c_2(t)$ (variable) are the damping coefficients of the dashpots, and m is the mass of the body. Since the damper with a coefficient $c_2(t)$ is assumed to suffer aging effects, we need an assumption on its dynamics.

Assumption 3.8. *The damping coefficient* $c_2(t)$ *has the following dynamics*

$$\dot{c}_2 = -\varepsilon(\dot{z} + c_2), \tag{3.26}$$

where ε *is an small real number.*

We define a the third state $x = c_2$, and assume that there is a sensor in the ageing damper which gives as output $y = x + z_1$. Then, the full model becomes

$$\dot{z}_1 = z_2 - u, \tag{3.27a}$$
$$\dot{z}_2 = -\frac{kz_1}{m} - \frac{xz_2}{m} - \frac{c_1 z_2}{m} + \frac{xu}{m} + \frac{c_1 u}{m}, \tag{3.27b}$$
$$\dot{x} = -\varepsilon(z_2 + x), \tag{3.27c}$$
$$y = x + z_1, \tag{3.27d}$$

where $z \in \mathbb{R}^2$ correspond to the fast state variables, $x \in \mathbb{R}$ is the slow state of the system, and $\varepsilon > 0$ is the singular perturbation parameter defining the time-scale separation. We define the slow time-scale as $t_s = \varepsilon t$. Therefore, the system (3.27) in the new time scale is given by

$$\frac{dx}{dt_s} = -z_2 - x, \tag{3.28a}$$
$$\varepsilon\frac{dz_1}{dt_s} = z_2 - u, \tag{3.28b}$$
$$\varepsilon\frac{dz_2}{dt_s} = -\frac{kz_1}{m} - \frac{xz_2}{m} - \frac{c_1 z_2}{m} + \frac{xu}{m} + \frac{c_1 u}{m}. \tag{3.28c}$$

It follows that the reduced system is defined by

$$\frac{dx}{dt_s} = -u - x, \tag{3.29a}$$
$$y_s = x. \tag{3.29b}$$

Let $k = 50$, $c_1 = 40$ and $m = 5$ and consider a circle criterion observer given by [9] for the reduced system (3.29). We then use the observer to estimate the slow states of the full system (3.28). We present simulation results in Figure 3.3.

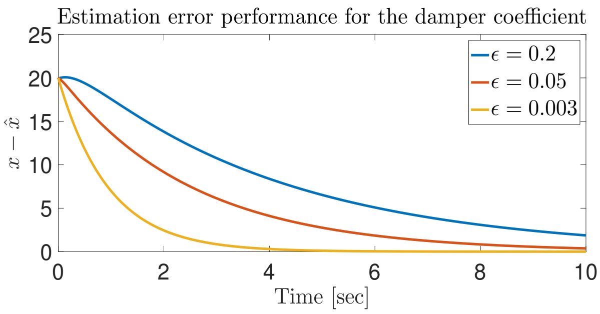

Figure 3.3: Estimation error performance of the damping properties of $c_2(t)$.

3.4 Circle criterion-based $\mathcal{H}_\infty$ observer for systems with linear output maps

In this section and the following one, we analyse the nonlinear observers introduced in [128]. Both observers and the class of plants for which they can be designed share some properties. The main difference between them is that the observer considered in this section deals with nonlinear systems with linear outputs while the observer in the next section considers nonlinear systems with nonlinear output maps. So, consider the class of plants with a singularly perturbed structure such that the slow model has a form for which results from [128] can be applied,

$$\dot{x} = Ax + G\gamma(x) + Bz, \tag{3.30a}$$

$$\varepsilon\dot{z} = M_1x + M_2z, \tag{3.30b}$$

$$y = C_1x + C_2z + Dw, \tag{3.30c}$$

where $x \in \mathbb{R}^n$ is the slow state, $z \in \mathbb{R}^m$ is the fast state, $y \in \mathbb{R}^p$ is the measured output variable, $w \in \mathbb{R}^q$ is the measurement noise which is $\mathcal{L}_2$ bounded, $\varepsilon > 0$ is the singular perturbation parameter of the process, $\gamma : \mathbb{R}^n \to \mathbb{R}^s$ is assumed to be globally Lipschitz and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $G \in \mathbb{R}^{n\times s}$, $C_1 \in \mathbb{R}^{p\times n}$, $C_2 \in \mathbb{R}^{p\times m}$, $D \in \mathbb{R}^{p\times q}$, $M_1 \in \mathbb{R}^{m\times n}$ and $M_2 \in \mathbb{R}^{m\times m}$. The function $\gamma(\cdot)$ has the form

$$\gamma(x) = \Big[\gamma_1(F_1x), \ldots, \gamma_i(F_ix), \ldots, \gamma_s(F_sx)\Big]^T, \tag{3.31}$$

where $F_i \in \mathbb{R}^{n_i\times n}$. The notation n_i represents the number of rows of F_i that are not constrained [128]. By having a linear fast dynamics, we guarantee that the reduced model has the appropriate structure for which we can design the nonlinear observer [128].

Assumption 3.9. *The matrix M_2 in (3.30b) is Hurwitz.*

The above assumption is essential since the approximation of the singularly perturbed system (3.30) through lower dimensional systems is only possible if M_2 is Hurwitz. Note that the system (3.30) does not have inputs, then Assumption 2.1 trivially holds. We now set $\varepsilon = 0$ in (3.30b), such that we obtain that $H(x) = -M_2^{-1}M_1x$. Hence, Assumption 2.2 holds. Note that the slow model is defined by

$$\dot{x} = A_0x + G\gamma(x), \tag{3.32a}$$

$$y_s = Cx + Dw, \tag{3.32b}$$

where $A_0 = A - BM_2^{-1}M_1$ and $C = C_1 - C_2M_2^{-1}M_1$.

Assumption 3.10. *The reduced system (3.32) is Input-to-State practically stable.*

By virtue of Assumption 3.10, there is a Lyapunov function such that the reduced system satisfies Assumption 2.3. This assumption gives generality to the plant since there is no need for A_0 to be Hurwitz. Now, define the change of variables $z = \xi - M_2^{-1}M_1x$. Therefore, we have

$$\dot{x} = A_0x + G\gamma(x) + B\xi, \tag{3.33a}$$

$$\varepsilon\dot{\xi} = M_2\xi + \varepsilon(M_2^{-1}M_1)[A_0x + G\gamma(x) + B\xi], \tag{3.33b}$$

$$y = Cx + C_2\xi + Dw \tag{3.33c}$$

Similarly to the previous sections, we have that the boundary layer system is defined by $d\xi/d\tau = M_2\xi$ where $\tau = t/\varepsilon$. So, we can check that Assumption 2.4 holds with $c_1 = \lambda_{\min}\{P_\xi\}$, $c_2 = \lambda_{\max}\{P_\xi\}$, $c_3 = -\lambda_{\min}\{Q_\xi\}$, $c_4 = 2\lambda_{\max}\{P_\xi\}$, $c_5 = 0$ and $c_6 = 0$.

The verification of the interconnection conditions in Assumptions 2.5 and 2.6 is done as in the previous case. The right-hand side of the system does not depend on ε, then $L_0 = 0$ and $L_3 = 0$. It follows from (3.32a) and (3.33a) that $L_1 = |B|$. The function $\gamma(\cdot)$ is globally Lipschitz and it is observed that it vanishes at $x = 0$, then there is a Lipschitz constant $L_\gamma > 0$ such that $|G\gamma(x)| \le L_\gamma|x|$. Moreover, $|A_0x| \le |A_0||x|$, so that $|A_0x + G\gamma(x)| \le (L_\gamma + |A_0|)|x|$. Then, $L_2 = L_\gamma + |A_0|$. Since the isolated solution $H(x)$ is a function of x we have $L_4 = 0$, $L_5 = |M_2^{-1}M_1|$ and $L_6 = 0$. Therefore, it follows that Assumptions 2.5 and 2.6 hold.

3.4.1 Observer design

We now consider the generalized circle criterion observer proposed in [128] for nonlinear systems with linear outputs. This observer is defined by

$$\dot{\hat{x}} = A_0\hat{x} + G\left[\gamma_1(\hat{v}_1), \ldots, \gamma_s(\hat{v}_s)\right]^T + L(y_s - C\hat{x}), \tag{3.34}$$

with $\hat{v}_i = F_i\hat{x} + K_i(y_s - C\hat{x})$ where $i \in \{1, \ldots, s\}$, $\hat{x} \in \mathbb{R}^n$ is the observer's state and an estimate of the slow state, K_i and L are gain matrices of appropriate dimensions which must be designed. We design the observer (3.34) for the reduced system (3.32). Define the estimation error as $e := x - \hat{x}$. By considering [Lemma 2, 128] and the features of the system such as the global Lipschitz property of $\gamma(\cdot)$ among other conditions, it is shown in [128] that the error dynamics is given by

$$\dot{e} = (A - LC + \mathbb{A})\, e + (-LD + \mathbb{B})\, w, \tag{3.35}$$

where $\mathbb{A} = \sum_{i,j=1}^{i,j=s,n_i} [\phi_{ij} G \mathcal{H}_{ij} \mathbb{F}_{K_i}]$ and $\mathbb{B} = \sum_{i,j=1}^{i,j=s,n_i} [\phi_{ij} G \mathcal{H}_{ij} \mathbb{D}_{K_i}]$, with $\mathcal{H}_{ij} = \hat{\iota}_s(i)\hat{\iota}_{n_i}^T(j)$, $\mathbb{F}_{K_i} = F_i - K_i C$, and $\mathbb{D}_{K_i} = -K_i D$, where $\hat{\iota}_s(i)$ is a vector of the canonical basis of $\mathbb{R}^s$ with the element 1 in the i-th position, the functions ϕ_{ij} are defined by [Lemma 2, 128] (see [128] for further details). The observer design problem is to determine the parameters such that the $\mathcal{H}_\infty$ criterion

$$|e|_{\mathcal{L}_2} \leq \rho |e_0| + \sqrt{\mu}|w|_{\mathcal{L}_2}, \tag{3.36}$$

is satisfied and μ is minimised. [Theorem 4, 128] gives a LMI that must be solved to minimize μ and to design the gain matrices of the observer. By solving the LMI, it is possible to construct the matrices L and K_i. Moreover, the solution of the LMI condition gives a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$ such that $V_2(e) = e^T P e$ is a Lyapunov function for (3.35). If P is a solution for the LMI in [Theorem 4, 128] , then its derivative satisfies

$$\frac{\partial V_2}{\partial e} f_e(e, w) \leq -|e|^2 + \mu |w|^2, \tag{3.37}$$

where $f_e(e, w)$ is given by the right-hand side of (3.35). It is observed that, by $\mathcal{L}_2$-norm, (3.37) implies that the $\mathcal{H}_\infty$ criterion is satisfied with $\rho = \lambda_{max}\{P\}$. Note that Assumption 2.7 holds with $a_1 = \lambda_{min}\{P\}$, $a_2 = \lambda_{max}\{P\}$, $a_3 = 1$, $a_4 = \mu$ and $a_5 = \lambda_{max}\{P\}$. We now look at the observer dynamics to verify Assumption 2.8. Note that the output of the system does not depend on ε then $L_7 = 0$. On the other hand, we have that $L_8 = |LC_2| + L_\gamma |G|\, |\sum_i K_i C_2|$ is such that (2.31) holds where $L_\gamma > 0$ is a Lipschitz constant for the function $\gamma(\cdot)$. Therefore, Assumptions 2.1 - 2.8 hold, and Theorem 2.1 holds too.

3.4.2 Simulation results

We now present a case study for the nonlinear observer described and analysed above. We consider an application to vehicle slip angle estimation. We consider the single track model for vehicle lateral dynamics described by Figure 3.4 which was introduced in [104]. The variables in Figure 3.4 define the following, F_{yf} and F_{yr} are the lateral tire forces of the front and rear wheels, F_{xf} and F_{xr} are the horizontal tire forces, a and b are the distances of the front and the rear tires from the centre of gravity of the vehicle, r is the yaw rate, β is the slip angle of the vehicle, and α_f and α_r are the tire slip angles of the front and the rear wheels.

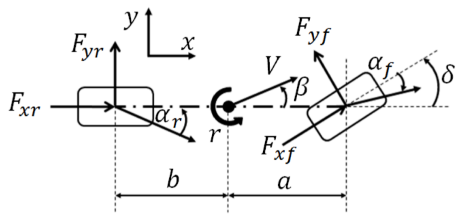

Figure 3.4: Single track model for vehicle lateral dynamics with linear output.

We assume an inertial measurement unit with fast linear dynamics is used to obtain the slip angle of the front tire. Hence, the nonlinear vehicle lateral dynamics with a fast sensor can be represented by the following system of equations

$$\begin{aligned}
\dot{x}_1 &= -\left(\frac{u_x}{a+b} + \frac{a^2 c_{1f}}{I_z u_x}\right) x_1 + \left(\frac{u_x}{a+b} + \frac{abc_{1r}}{I_z u_x}\right) x_2 - \frac{a^2}{I_z u_x}\gamma_1(x_1) \\
&\quad + \frac{ab}{I_z u_x}\gamma_2(x_2) + \frac{u_x}{a+b}\delta + \dot{\delta} - \frac{1}{u_x}\alpha_y, && (3.38a) \\
\dot{x}_2 &= -\left(\frac{u_x}{a+b} - \frac{abc_{1f}}{I_z u_x}\right) x_1 + \left(\frac{u_x}{a+b} - \frac{b^2 c_{1r}}{I_z u_x}\right) x_2 + \frac{ab}{I_z u_x}\gamma_1(x_1) \\
&\quad - \frac{b^2}{I_z u_x}\gamma_2(x_2) + \frac{u_x}{a+b}\delta - \frac{1}{u_x}\alpha_y, && (3.38b) \\
\varepsilon \dot{z} &= x_1 - z, && (3.38c) \\
y &= z_1. && (3.38d)
\end{aligned}$$

where $x_1 = \alpha_f$, $x_2 = \alpha_r$, u_x is the longitudinal velocity, c_i are the tire coefficients, I_z is the vehicle inertial, δ is the steering angle, α_y is the lateral acceleration, and

$$\begin{aligned}
\gamma_1(x_1) &= -c_2 x_1^2 \mathrm{sign}(x_1) + c_3 x_1^3, \\
\gamma_2(x_2) &= -c_2 x_2^2 \mathrm{sign}(x_2) + c_3 x_2^3.
\end{aligned}$$

We need to estimate both front and rear tire slip angles so that one can compute the slip angle of the vehicle. This can be done, for example, by using the formula $\beta = -\alpha_r -$

rb/u_x. Hence, we compute the reduced order model which is given as follows

$$\dot{x}_1 = -\left(\frac{u_x}{a+b} + \frac{a^2 c_{1f}}{I_z u_x}\right) x_1 + \left(\frac{u_x}{a+b} + \frac{abc_{1r}}{I_z u_x}\right) x_2 - \frac{a^2}{I_z u_x}\gamma_1(x_1)$$
$$+ \frac{ab}{I_z u_x}\gamma_2(x_2) + \frac{u_x}{a+b}\delta + \dot{\delta} - \frac{1}{u_x}\alpha_y, \tag{3.39a}$$
$$\dot{x}_2 = -\left(\frac{u_x}{a+b} - \frac{abc_{1f}}{I_z u_x}\right) x_1 + \left(\frac{u_x}{a+b} - \frac{b^2 c_{1r}}{I_z u_x}\right) x_2 + \frac{ab}{I_z u_x}\gamma_1(x_1)$$
$$- \frac{b^2}{I_z u_x}\gamma_2(x_2) + \frac{u_x}{a+b}\delta - \frac{1}{u_x}\alpha_y, \tag{3.39b}$$
$$y = x_1. \tag{3.39c}$$

Since the reduced order system (3.39) can be written in the form of (3.32), we are able to use the proposed approach in [128] to construct an observer for the reduced system. By following the methodology given in [128], we obtain the gain matrices for the observer in (3.34), which are given by

$$L = \begin{bmatrix} 0.161 & 0.168 \\ 0.052 & 0.616 \end{bmatrix}, \quad K = \begin{bmatrix} -0.975 & 0.015 \end{bmatrix}.$$

We performed simulations with different values of the perturbation parameter to test how the system behaves when the sensor has different time-constant. The simulation results for this numerical example are displayed in Figure 3.5.

3.5 Circle criterion-based $\mathcal{H}_\infty$ observer for systems with nonlinear output maps

We now consider nonlinear systems with reduced order models with nonlinear outputs for which we can design the observer introduced in [128]. Consider the following class of plants with a singularly perturbed structure

$$\dot{x} = Ax + G\gamma(x) + B_1 z, \tag{3.40a}$$
$$\varepsilon\dot{z} = M_1 x + M_2 z, \tag{3.40b}$$
$$y = Cx + B_2\tilde{\gamma}(x, z) + Dw, \tag{3.40c}$$

where $x \in \mathbb{R}^n$ is the slow state, $z \in \mathbb{R}^m$ is the fast state, $y \in \mathbb{R}^p$ is the measured output variable, $w \in \mathbb{R}^q$ is the measurement noise which is $\mathcal{L}_2$ bounded, $\varepsilon > 0$ is the singular

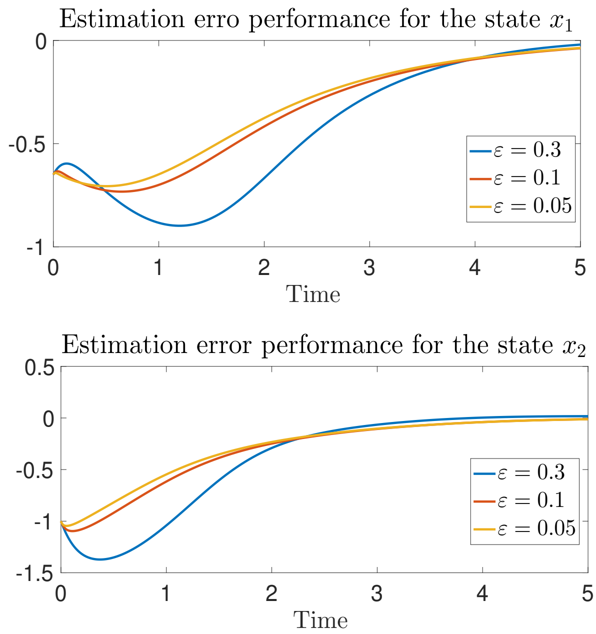

Figure 3.5: Simulations results for automotive slip angle estimation with nonlinear output.

perturbation parameter of the process, the maps $\gamma : \mathbb{R}^n \to \mathbb{R}^s$ and $\tilde{\gamma} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^r$ are assumed to be globally Lipschitz and $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m}$, $B_2 \in \mathbb{R}^{p\times r}$, $G \in \mathbb{R}^{n\times s}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times q}$, $M_1 \in \mathbb{R}^{m\times n}$ and $M_2 \in \mathbb{R}^{m\times m}$. The function $\gamma(\cdot)$ has the form given in (3.31).

Assumption 3.11. *The matrix* M_2 *in (3.40b) is Hurwitz.*

The above assumption is essential since the approximation of the singularly perturbed system (3.40) through lower dimensional systems is only possible if Assumption 3.11 holds. Since the plant (3.40) does not have inputs, Assumption 2.1 trivially holds. Moreover, Assumption 2.2 holds by virtue of Assumption 3.11. Note that the isolated solution is given by $H(x) = -M_2^{-1}M_1x$ and the slow model by

$$\dot{x} = A_0x + G\gamma(x), \tag{3.41a}$$

$$y_s = Cx + B_2\overline{\gamma}(x) + Dw, \tag{3.41b}$$

where $A_0 = A - B_1M_2^{-1}M_1$ and $\overline{\gamma}(x) = \tilde{\gamma}(x, -M_2^{-1}M_1x)$. It is assumed that the function $\overline{\gamma}(\cdot)$ has the following form

$$\overline{\gamma}(x) = \left[\overline{\gamma}_1(E_1x), \ldots, \overline{\gamma}_i(E_ix), \ldots, \overline{\gamma}_r(E_rx)\right]^T, \tag{3.42}$$

where $E_i \in \mathbb{R}^{p_i \times n}$. The notation p_i represents the number of rows of E_i are not constrained, see [128] for further details.

Assumption 3.12. *The reduced system (3.41) is Input-to-state practically stable.*

Assumption 2.3 holds by virtue of Assumption 3.12. Now, we perform the change of variables $z = \xi - M_2^{-1}M_1x$ so that we obtain

$$\dot{x} = A_0x + G\gamma(x) + B_1\xi, \tag{3.43a}$$

$$\varepsilon\dot{\xi} = M_2\xi + \varepsilon(M_2^{-1}M_1)[A_0x + G\gamma(x) + B_1\xi], \tag{3.43b}$$

$$y = Cx + B_2\tilde{\gamma}(x, \xi - M_2^{-1}M_1x) + Dw. \tag{3.43c}$$

Observe that (3.43) leads to the boundary layer system given by $d\xi/d\tau = M_2\xi$ where $\tau = t/\varepsilon$. It follows that there are two positive definite symmetric matrices Q_ξ and P_ξ satisfying (3.7) so that $W(\xi) = \xi^T P_\xi \xi$ is a candidate Lyapunov function for the boundary layer system. Therefore, Assumption 2.4 holds with $c_1 = \lambda_{\min}\{P_\xi\}$, $c_2 = \lambda_{\max}\{P_\xi\}$, $c_3 = -\lambda_{\min}\{Q_\xi\}$, $c_4 = 2\lambda_{\max}\{P_\xi\}$, $c_5 = 0$ and $c_6 = 0$.

The verification of the interconnection conditions in Assumptions 2.5 and 2.6 is done as in the previous case. Then, we have that $L_0 = 0$, $L_1 = |B_1|$, $L_2 = L_\gamma + |A_0|$, $L_3 = 0$, $L_4 = 0$, $L_5 = |M_2^{-1}M_1|$ and $L_6 = 0$ where $L_\gamma > 0$ is such that $|G\gamma(x)| \leq L_\gamma|x|$. Therefore, it follows that Assumptions 2.5 and 2.6 hold.

3.5.1 Observer design

Consider the circle criterion-based $\mathcal{H}_\infty$ observer for nonlinear systems with nonlinear output maps introduced in [128]. The dynamics of the observer is given by

$$\dot{\hat{x}} = A_0\hat{x} + G\left[\gamma_1(\hat{v}_1), \ldots, \gamma_s(\hat{v}_s)\right]^T + L\left(y_s - C\hat{x} - B_2\left[\overline{\gamma}_1(\hat{\theta}_1), \ldots, \gamma_r(\hat{\theta}_r)\right]^T\right) \tag{3.44}$$

with

$$\hat{v}_i = F_i\hat{x} + K_i\left(y_s - C\hat{x} - B_2\left[\overline{\gamma}_1(\hat{\theta}_1), \ldots, \gamma_r(\hat{\theta}_r)\right]^T\right),$$
$$\hat{\theta}_i = E_i\hat{x} + N_i\left(y_s - C\hat{x} - B_2\left[\overline{\gamma}_1(E_1\hat{x}), \ldots, \gamma_r(E_r\hat{x})\right]^T\right),$$

for $i \in \{1, \ldots, m\}$, $\hat{x} \in \mathbb{R}^n$ is the observer's state and an estimate of the slow state, $K_i \in \mathbb{R}^{n_i \times p}$, $L \in \mathbb{R}^{n \times p}$ and $N_i \in \mathbb{R}^{p_i \times p}$ are the gain matrices of the observer which must be designed. By following the approach described in Chapter 2, we design the observer (3.44) for the reduced system (3.41).

We now define the estimation error as $e := x - \hat{x}$ so that we have from [128] that the error dynamics are defined by

$$\dot{e} = (A - LC + \mathbb{A})\, e + \mathbb{C}e + (-LD + \mathbb{B})\, w + \mathbb{D}w \tag{3.45}$$

where $\mathbb{A} = \sum_{i,j=1}^{i,j=s,n_i} [\phi_{ij} G\mathcal{H}_{ij}\mathbb{F}_{K_i}]$, $\mathbb{B} = \sum_{i,j=1}^{i,j=s,n_i} [\phi_{ij} G\mathcal{H}_{ij}\mathbb{J}_{K_i}]$, $\mathbb{C} = \sum_{i,j=1}^{i,j=r,p_i} [\psi_{ij} LB_2\mathcal{F}_{ij}\mathbb{E}_{N_i}]$ and $\mathbb{D} = \sum_{i,j=1}^{i,j=r,p_i} [\psi_{ij} LB_2\mathcal{F}_{ij}\mathbb{J}_{N_i}]$, for which $\mathcal{H}_{ij} = \hat{\iota}_s(i)\hat{\iota}_{n_i}^T(j)$, $\mathbb{F}_{K_i} = F_i - K_iC$, $\mathbb{J}_{K_i} = -K_iD$, $\mathbb{E}_{N_i} = N_iC - E_i$ and $\mathbb{J}_{N_i} = N_iD$, where $\hat{\iota}_s(i)$ is a vector of the canonical basis of $\mathbb{R}^s$ with the element 1 in the ith position, the functions ϕ_{ij} and ψ_{ij} are defined by [Lemma 2, 128] (see [128] for further details). Similarly to the previous section, the observer design must be performed such that the $\mathcal{H}_\infty$ criterion in (3.36) holds while μ is minimized. [Theorem 5, 128] states that the optimization problem of minimizing μ is solved by obtaining a solution to an LMI. Then, with the solution of the LMI one can construct the gain matrices L, K_i and N_i. Furthermore, the solution of the LMI condition generates a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that $V_2(e) = e^TPe$ is a Lyapunov function for (3.45). Then, it follows from [Theorem 5, 128], that

$$\frac{\partial V_2}{\partial e} f_e(e, w) \leq -|e|^2 + \mu|w|^2 \tag{3.46}$$

where $f_e(e, w)$ is given by the right-hand side of (3.45). Then, (3.46) implies that the $\mathcal{H}_\infty$ criterion (3.36) is satisfied with $\rho = \lambda_{max}\{P\}$. Note that Assumption 2.7 holds with $a_1 = \lambda_{min}\{P\}$, $a_2 = \lambda_{max}\{P\}$, $a_3 = 1$, $a_4 = \mu$ and $a_5 = \lambda_{max}\{P\}$. Since the output of the system does not depend on ε then $L_7 = 0$ in Assumption 2.8. On the other hand, we have that $L_8 = L_{\tilde{\gamma}}|LB_2||\,|I + \sum_i N_iB_2| + L_\gamma L_{\tilde{\gamma}}|G|\,|\sum_i K_iB_2| + L_\gamma L_{\tilde{\gamma}}^2|G||B_2|\,|\sum_i N_iB_2|$ is such that (2.31) holds where $L_\gamma > 0$ and $L_{\tilde{\gamma}} > 0$ are Lipschitz constants for functions $\gamma(\cdot)$ and $\tilde{\gamma}(\cdot,\cdot)$. Therefore, Assumptions 2.1 - 2.8 hold, and Theorem 2.1 holds too.

3.5.2 Simulation results

We now consider the single track model for vehicle lateral dynamics from Section 3.4.2 which is described by the system of differential equations (3.38). However, in this section, the output of the system is given by

$$y_1 = -\left(\frac{u_x}{a+b}\right)x_1 + \left(\frac{u_x}{a+b}\right)x_2, \tag{3.47a}$$

$$y_2 = -\left(\frac{c_{1f}}{m}\right)x_1 + \left(\frac{c_{1r}}{m}\right)x_2 + \frac{\gamma_1(x_1)}{m} + \frac{\gamma_2(x_2)}{m}. \tag{3.47b}$$

By considering this nonlinear output, we design an observer in the form of (3.44) to estimate the slow variables of the system (3.38) with nonlinear output (3.47). By following the proposed approach in [128], we obtain that the gain matrices are given as follows

$$L = \begin{bmatrix} 0.069 & 0.247 \\ 0.041 & 0.385 \end{bmatrix}, \quad K_1 = \begin{bmatrix} -0.075 & 0.009 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.075 & 0.009 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} -0.065 & 0.012 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0.065 & 0.012 \end{bmatrix}.$$

In Figure 3.6, we present simulation results for different values of $\varepsilon > 0$. It can be seen that the estimation error performs better when the perturbation parameter is smaller.

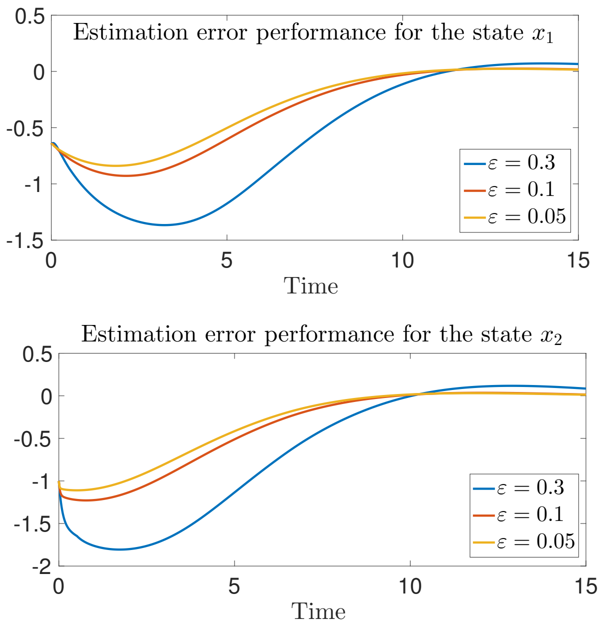

Figure 3.6: Simulations results for automotive slip angle estimation.

3.6 Conclusions of the chapter

We have reported in a conference paper that results from Chapter 2 apply to the class of systems and the nonlinear observer considered in [68] when we assume global Lipschitz conditions for the reduced order model. Note that assumptions in [24] imply our conditions hold such that results in [24] are covered by our findings in Chapter 2. Our conclusions also cover those cases when results in [123] are used only for the estimation of the slow states of the plant. Here, we presented four classes of systems and observers that are not covered by the existing literature.

We demonstrated that the estimation framework developed in Chapter 2 covers at least four classes of globally Lipschitz nonlinear singularly perturbed systems and four nonlinear full-order observers that can be designed for the reduced order models of those systems. We stated how those classes of plants and observers satisfy the given assumptions in Chapter 2. Furthermore, we illustrated the applicability of the theoretical framework by presenting simulation results for each observer. We presented two classes of observers for systems with outputs disturbed by measurement noise and proved that they can be used to estimate the slow states of a globally Lipschitz nonlinear singularly perturbed plant. This is a significant outcome of this chapter and Chapter 2 since, as far as we are aware, the estimation problem of globally Lipschitz nonlinear singularly perturbed systems in the presence of measurement noise has not been addressed before.

Part II

Observers of General Dimension for the Slow State Estimation of Nonlinear Singularly Perturbed Systems

Introduction to Part II

IN THIS part of the thesis, we present semi-global stability results for the estimation error of nonlinear observers used to estimate the slow state of a more general class of nonlinear singularly perturbed systems than those considered in Part I. We tackle the slow state estimation problem represented by the block diagram in Figure II.1. We deal with observers of general dimension, i.e. full-order, reduced-order and higher-order estimators. Similarly to Part I, we assume that measurement noise is a disturbance to the output of the system. Although we use standard singular perturbation techniques, we address a problem that has not been considered in the literature before. Furthermore, we develop a design framework with such generality that our assumptions hold for many classes of nonlinear systems and observers. We demonstrate that when our assumptions are relaxed to hold locally, existing results in [68] are covered by our framework.

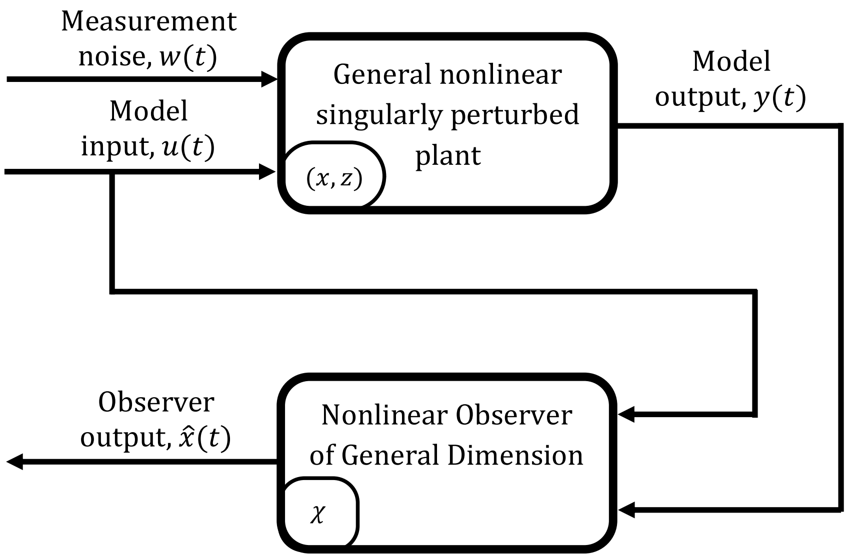

Figure II.1: Slow state estimation via observers of general dimension for nonlinear singularly perturbed systems.

In Chapter 4, we introduce the theoretical results that solve the problem depicted by Figure II.1. We present semi-global practical input-to-state stability results for the estimation error dynamics as well as $\mathcal{L}_\infty \cap \mathcal{L}_2$ stability results when the measurement noise belongs to $\mathcal{L}_\infty \cap \mathcal{L}_2$, i.e. $w \in \mathcal{L}_\infty \cap \mathcal{L}_2$. Our results lead to semi-global practical asymptotical (SPA) stability of the error dynamics in the absence of measurement noise. We also present a boundedness of solutions result for the singularly perturbed plant as part of the theoretical body developed in this part of the thesis.

The theoretical results in Chapter 4 constitute a design framework for slow state estimation of nonlinear singularly perturbed systems. Hence, we study three classes of systems and nonlinear observers in Chapter 5 to demonstrate the applicability of our conclusions. Furthermore, we demonstrate how the class of nonlinear systems and the nonlinear observer considered in [68] satisfy our assumptions. We also present numerical examples and simulation results to illustrate our findings.

Chapter 4

Semi-Global Stability of Nonlinear Observers for the Estimation of the Slow States

In this chapter, we generalise existing results on the slow state estimation for systems with two time-scales. We focus on estimating only the slow variables of the system by assuming that the output and the input are available. Our results consider boarder classes of plants and estimators than results in Chapter 2 and other results in the literature. Furthermore, we deal with observers of general dimension, i.e. we cover reduced-order, full-order and higher-order estimators. We study the robustness of the estimators to singular perturbations and to measurement noise since we consider systems with outputs corrupted by measurement noise. Here, we present results that lead to semi-global practical asymptotical (SPA) stability in the absence of measurement noise.

4.1 Introduction

TO GENERALISE findings in [68] and Chapter 2, we now consider a broader class of systems and nonlinear observers of general dimension, i.e. reduced-order, full-order and higher-order observers. We deal with singularly perturbed systems where the measured output is corrupted by measurement noise. Our goal is to analyse the robustness to singular perturbations and measurement noise of nonlinear observers designed to estimate the slow states of a nonlinear singularly perturbed system. We do not consider the fast variables for the observer design process; instead, the observer is synthesised for the reduced (slow) model. We work with this approach since model-based observer design for singularly perturbed systems may lead to ill-conditioned observer gains, and subsequently, to undesired convergence properties of the estimation error if the observer is designed for the full plant.

Although we use a standard singular perturbation approach, we address an estimation problem that was not previously considered in the literature. Then, we generate a general estimation framework for nonlinear singularly perturbed systems in the standard form that was missing in the literature. We prove that, under a set of general assumptions on the observer and the plant, the estimation error is semi-globally input-to-state practically stable where the slow and fast states are seen as inputs. In our proofs, we take advantage of the cascade properties of the observer and error dynamics. Moreover, we prove that the input-to-state property of the estimation error dynamics leads to semi-global practical asymptotical stability in the perturbation parameter in the absence of measurement noise.

Since we consider nonlinear plants with outputs disturbed by measurement noise, we also provide $\mathcal{L}_2 \cap \mathcal{L}_\infty$ stability results to cover cases where the noise belongs to $\mathcal{L}_2 \cap \mathcal{L}_\infty$. Furthermore, as far as we are aware, there are no existing results in the literature dealing with measurement noise in the estimation context of nonlinear singularly perturbed systems. To conclude the input-to-state stability and $\mathcal{L}_2 \cap \mathcal{L}_\infty$ results, we first prove that, under certain assumptions on the reduced and boundary layer systems, the singularly perturbed plant exhibits an input-to-state practical stability property with respect to the input and its derivative (practical DISS) as well as a practical $\mathcal{L}_2$ stability as defined in [Property I3, 95].

This chapter is organized as follows. Section 4.2 introduces the plant and assumptions placed upon it. Section 4.3 demonstrates boundedness of solutions of the original system (Lemma 4.1) and SPA stability for the fast states of the system (Corollary 4.1). Section 4.4 contains the main result of this chapter. First, boundedness of solutions is proven for the observer dynamics (Corollary 4.2). Then, a semi-global practical ISS and $\mathcal{L}_2$ stability properties are proven for the error dynamics in our main contribution (Theorem 4.1). Section 4.5 presents the conclusions of the chapter.

4.2 General setting

In this section, we introduce a general class of nonlinear systems as well as the lower dimensional systems: the reduced order and the boundary layer systems. Moreover, we state a set of appropriate assumptions that allow us to analyse the performance of the singularly perturbed plant through the lower dimensional systems. In this work, we follow the traditional approach of singular perturbations theory but applied to systems with outputs as they are required by the problem we address here. Consider the general class of plants in the following singularly perturbed form

$$\dot{x} = f_s(t, x, z, u(t), \varepsilon), \tag{4.1a}$$

$$\varepsilon\dot{z} = f_f(t, x, z, u(t), \varepsilon), \tag{4.1b}$$

$$y = h(t, x, z, u(t), w(t), \varepsilon), \tag{4.1c}$$

where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$ represent the slow and fast states of the system respectively, $y \in \mathbb{R}^p$ is the measured output, $\varepsilon > 0$ is the singular perturbation parameter representing the time scale separation, $u \in \mathbb{R}^r$ denotes the input vector and $w \in \mathbb{R}^s$ is the measurement noise to the system which is assumed to be bounded. The vector $u(t)$ is a measured input that may represent a control input, exogenous measured disturbances, constant or time-varying parameters or tracking signals. In the sequel, for simplicity, we will suppress the argument t in the notation of the vector input $u(t)$ and in the measurement noise $w(t)$. Observe that the estimation problem we address in this chapter leads to weaker results than Chapter 2 since here we consider weaker assumptions on the stability properties of the reduced order system and the boundary layer model. Although results in here and in Chapter 2 are related, they do not imply each other.

Assumption 4.1. *The input of the system u is differentiable and its derivative is bounded uniformly in ε for $\varepsilon \ll 1$. In addition, the input, its derivative and the measurement noise belong to $\mathcal{L}_\infty$; i.e., $u, \dot{u}, w \in \mathcal{L}_\infty$.*

Singular perturbations techniques are useful to reduce the complexity of the problem by taking advantage of the time-scale separation of the system. The standard singular perturbations technique is the decomposition of original system (4.1) into lower dimensional systems associated with different time scales which in general are easier to deal with. Then, Assumption 4.1 is helpful when we intend to prove a result by working with the reduced order system and the boundary layer system.

By following the standard singular perturbations technique, we set $\varepsilon = 0$ so that we restrict the performance of the system to the slow manifold given by the following algebraic equation

$$0 = f_f(t, x, z, u, 0). \tag{4.2}$$

Assumption 4.2. *The algebraic equation (4.2) has an isolated solution $z = H(t, x, u)$ that can be obtained analytically and is used to define the reduced (slow) system.*

Assumption 4.2 is common in the singular perturbations framework since it is required to analyse the quasi-steady state behaviour of the singularly perturbed system. Moreover, we must know $H(t,x,u)$ to define the slow system which is needed to designing an observer for the slow states of the plant. Note that one can work with an approximation of $H(t,x,u)$ which would open a new area for further research. Since we assume that we know $H(t,x,u)$, we substitute the isolated solution $z = H(t,x,u)$ in (4.1a) and (4.1c) at $\varepsilon = 0$ to obtain the reduced (slow) dynamical system

$$\dot{x} = f_s(t,x,H(t,x,u),u,0), \tag{4.3a}$$

$$y_s = h(t,x,H(t,x,u),u,w,0). \tag{4.3b}$$

To be able to conclude a result for the singularly perturbed plant (4.1), we need to assume certain properties on the reduced system (4.3). Hence, consider the following stability property for the reduced system.

Assumption 4.3. *For the slow system (4.3), there exists a continuously differentiable function* $V_1(t,x)$, *class-*$\mathcal{K}_\infty$ *functions* $\underline{\alpha}_{V_1}(\cdot)$, $\overline{\alpha}_{V_1}(\cdot)$, $\alpha_{V_1}(\cdot)$, $\gamma_{V_1}(\cdot)$, *and* $\zeta_1 > 0$, $\delta_{V_1} \geq 0$, *such that for all* $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$, $t \geq 0$

$$\underline{\alpha}_{V_1}(|x|) \leq V_1(t,x) \leq \overline{\alpha}_{V_1}(|x|), \tag{4.4}$$

$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x} f_s(t,x,H(t,x,u),u,0) \leq -\zeta_1 \alpha_{V_1}^2(|x|) + \gamma_{V_1}(|u|) + \delta_{V_1}. \tag{4.5}$$

Note that Assumption 4.3 implies that, for any bounded input, the system (4.3) is globally input-to-state practically stable [33, 114, 116]. This assumption is standard in nonlinear systems since the observer design for nonlinear unbounded systems is notoriously difficult. It covers systems with globally stable limit cycles; for instance, the Van der Pol Oscillator [70], Hamiltonian systems [22], the elastic pendulum [46], and so on. To analyse the fast dynamics behaviour, we consider the change of variables $\xi = z - H(t,x,u)$. The system (4.1) in the new coordinates (x,ξ) is represented by

$$\dot{x} = f_s(t,x,\xi + H(t,x,u),u,\varepsilon), \tag{4.6a}$$

$$\begin{aligned}\varepsilon\dot{\xi} = {} & f_f(t,x,\xi + H(t,x,u),u,\varepsilon) - \varepsilon\frac{\partial H}{\partial t} - \varepsilon\frac{\partial H}{\partial u}\dot{u} \\ & - \varepsilon\frac{\partial H}{\partial x} f_s(t,x,\xi + H(t,x,u),u,\varepsilon),\end{aligned} \tag{4.6b}$$

$$y = h(t,x,\xi + H(t,x,u),u,w,\varepsilon), \tag{4.6c}$$

in which the quasi-steady-state of the fast dynamics is $\xi = 0$. Consider the fast time scale τ defined as $\tau := \frac{t-t_0}{\varepsilon}$. Hence, in the τ-time scale, the singularly perturbed system (4.6a)-(4.6b) takes the form

$$\frac{dx}{d\tau} = \varepsilon f_s(t, x, \xi + H(t, x, u), u, \varepsilon), \tag{4.7a}$$

$$\begin{aligned}\frac{d\xi}{d\tau} &= f_f(t, x, \xi + H(t, x, u), u, \varepsilon) - \varepsilon\frac{\partial H}{\partial t} - \varepsilon\frac{\partial H}{\partial u}\dot{u} \\ &\quad - \varepsilon\frac{\partial H}{\partial x} f_s(t, x, \xi + H(t, x, u), u, \varepsilon).\end{aligned} \tag{4.7b}$$

Setting $\varepsilon = 0$ freezes the variables $t = t_0$ and $x = x(t_0)$, and reduces (4.7b) to the autonomous system

$$\frac{d\xi}{d\tau} = f_f(t_0, x(t_0), \xi + H(t_0, x(t_0), u), u, 0). \tag{4.8}$$

Observe that the solutions of (4.8) will converge to an $O(\varepsilon)$ neighbourhood of the origin during the boundary layer interval, see [70]. After that interval, the slowly varying parameters (t, x) are not longer close enough to their initial values $(t_0, x(t_0))$. Then, a stability property must be assumed for (4.8) such that its solutions remain in a neighbourhood of zero. To do so, the frozen variables $t = t_0$ and $x = x(t_0)$ must be allowed to take values in the region of the the slowly varying parameters (t, x). Therefore, we rewrite (4.8) as follows

$$\frac{d\xi}{d\tau} = f_f(t, x(t), \xi + H(t, x(t), u), u, 0), \tag{4.9}$$

where (t, x) are thought as fixed parameters. We refer to (4.9) as the boundary layer system. For further details on how the boundary layer system is obtained, the reader can refer to [Chapter 11, 70] and/or [Chapter 7, 75].

Assumption 4.4. *For the Boundary Layer System (4.9) there exists a Lyapunov function* $W(t, x, \xi)$ *and class-*$\mathcal{K}_\infty$ *functions* $\underline{\alpha}_W(\cdot)$, $\overline{\alpha}_W(\cdot)$ *and* $\alpha_W(\cdot)$, *and* $\zeta_3 > 0$ *such that for all* t, x, ξ *we have*

$$\underline{\alpha}_W(|\xi|) \leq W(t, x, \xi) \leq \overline{\alpha}_W(|\xi|), \tag{4.10}$$

$$\frac{\partial W}{\partial \xi} f_f(t, x, \xi + H(t, x, u), u, 0) \leq -\zeta_3 \alpha_W^2(|\xi|). \tag{4.11}$$

Note that Assumption 4.4 implies that the boundary layer dynamics are globally asymptotically stable uniformly in t, x and u. The above assumption is standard in the singular perturbation literature [33, 70, 75], and is critical in justifying the model reduction.

4.3 Boundedness of solutions of the plant

In this section, we provide a result that characterises the boundedness of solutions of the system (4.6) under general conditions, with a view to later using it for robustness analysis of the proposed approach to observer design. Although boundedness of solutions of the full system can be assumed, we prove it for two reasons, 1) this result is of interest in its own right and 2) some of the assumptions we state for this result are also needed to prove much stronger conclusion on the stability of the error dynamics. In our analysis, we compute the derivatives of $V_1(t, x)$ and $W(t, x, \xi)$, given in Assumptions 4.3 and 4.4, along the trajectories of (4.6). This leads to some terms representing the interconnections between the slow and the fast dynamics. In general, those interconnection terms are sign indefinite; therefore, we need appropriate conditions to bound them to conclude boundedness of solutions. These interconnection conditions were carefully chosen so that they cover as many classes of systems as possible. We demonstrate in Chapter 5 that a set of examples satisfies these conditions. Hence, our results provide a solid observer design framework for nonlinear singularly perturbed systems.

Assumption 4.5. *Consider $\alpha_{V_1}(\cdot)$ and $\alpha_W(\cdot)$ given in Assumptions 4.3 and 4.4 respectively. Suppose there exist non-negative constants a_i ($i = 1, 2, 3$) and b_i ($i = 1, 2, 3$), and class-$\mathcal{K}_\infty$ functions $\gamma_i(\cdot)$ ($i = 1, \dots, 4$), so that the following conditions hold*

$$\left|\frac{\partial V_1}{\partial x}[f_s(t, x, \xi + H(t, x, u), u, \varepsilon) - f_s(t, x, H(t, x, u), u, 0)]\right| \leq \varepsilon a_1 \alpha_{V_1}^2(|x|) + \varepsilon\gamma_1(|u|)\alpha_{V_1}(|x|) + b_1\alpha_{V_1}(|x|)\alpha_W(|\xi|), \tag{4.12}$$

$$\left|\frac{\partial W}{\partial \xi}[f_f(t, x, \xi + H(t, x, u), u, \varepsilon) - f_f(t, x, \xi + H(t, x, u), u, 0)]\right| \leq \varepsilon a_2 \alpha_W^2(|\xi|) + \varepsilon\gamma_2(|u|)\alpha_W(|\xi|) + \varepsilon b_2\alpha_{V_1}(|x|)\alpha_W(|\xi|), \tag{4.13}$$

$$\left|\frac{\partial W}{\partial t} - \frac{\partial W}{\partial \xi}\frac{\partial H}{\partial t} - \frac{\partial W}{\partial \xi}\frac{\partial H}{\partial u}\dot{u} + \left[\frac{\partial W}{\partial x} - \frac{\partial W}{\partial \xi}\frac{\partial H}{\partial x}\right] f_s(t, x, \xi + H(t, x, u), u, \varepsilon)\right| \leq a_3\alpha_W^2(|\xi|) + b_3\alpha_{V_1}(|x|)\alpha_W(|\xi|) + \gamma_3(|u|)\alpha_W(|\xi|) + \gamma_4(|\dot{u}|)\alpha_W(|\xi|), \tag{4.14}$$

for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^m$, $u \in \mathbb{R}^r$, $\dot{u} \in \mathbb{R}^r$ *and* $t \geq 0$.

Remark 4.1. *Assumption 4.5 can be relaxed to hold regionally or locally. Moreover, conditions in Assumption 4.5 can be relaxed to hold semi-globally with respect to the perturbation parameter* ε*. This means that for any positive constants* $\delta_1 > 0$, $\delta_2 > 0$, $\delta_3 > 0$ *there exists* $\varepsilon^*_{a_5} > 0$ *such that (4.12) - (4.14) holds for all* $\varepsilon \in (0, \varepsilon^*_{a_5})$ *and for all* $|(x, \xi)| \leq \delta_1$, $|u| \leq \delta_2$, $|\dot{u}| \leq \delta_2$ *and* $t \geq 0$*. Our proofs are such that our results can be easily extended to cover these cases.*

The inequalities (4.12) - (4.14) are general and similar to the ones in [74]. These interconnection conditions are satisfied in a number of real world examples we considered; for instance, a suspension system [70], a biological reactor [68], a three-state SCR catalyst [120], and so on. Note that we have also checked classes of plants for which these interconnection conditions hold. For example, the class of systems covered by the circle criterion observer [9] and the class of plants in the observability canonical form [14]. Moreover, the above inequalities can be verified in several examples by using quadratic-type Lyapunov functions [106]. A set of examples that satisfies Assumption 4.5 is presented in the next chapter.

We now present our first result of this chapter (Lemma 4.1) which states that, under Assumptions 4.1 - 4.5, for sufficiently small values of ε, the singularly perturbed system (4.6) exhibits a practical input-to-state stability with respect to the input and its derivative. Moreover, we also present a result in terms of $\mathcal{L}_2$ stability which guarantees that bounded energy inputs imply practical bounded solutions. Results in Lemma 4.1 are used later to prove the main result of the chapter.

Lemma 4.1. *Consider the singularly perturbed system (4.6). If Assumptions 4.1 - 4.5 hold, there exists a composite Lyapunov function* $V(t, x, \xi)$*, class-*$\mathcal{K}_\infty$ *functions* $\underline{\alpha}_V(\cdot)$, $\overline{\alpha}_V(\cdot)$, $\gamma_V(\cdot)$, $\tilde{\gamma}_V(\cdot)$, $\hat{\gamma}_V(\cdot)$*, and* $\mu_V > 0$*, such that there exists* $\tilde{\varepsilon}^* > 0$ *and* $\alpha_V(\cdot) \in \mathcal{K}_\infty$*, such that*

$$\underline{\alpha}_V(|(x, \xi)|) \leq V(t, x, \xi) \leq \overline{\alpha}_V(|(x, \xi)|), \tag{4.15}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f_s + \frac{\partial V}{\partial \xi} f_f \leq -\alpha_V(|(x, \xi)|) + \gamma_V(|u|) + \varepsilon\tilde{\gamma}_V(|u|) + \varepsilon\hat{\gamma}_V(|\dot{u}|) + \mu_V, \tag{4.16}$$

hold for all $\varepsilon \in (0, \tilde{\varepsilon}^*)$ *and for all* $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^m$, $u \in \mathbb{R}^r$, $\dot{u} \in \mathbb{R}^r$ *and* $t \geq 0$*. Consequently, there exists* $\beta_{L_1}(\cdot, \cdot) \in \mathcal{KL}$, $\gamma_{L_1}(\cdot) \in \mathcal{K}_\infty$*, class-*$\mathcal{K}_\infty$ *functions* $\tilde{\gamma}_\varepsilon(\cdot)$, $\hat{\gamma}_\varepsilon(\cdot)$

parametrized by ε *(their argument is of order* $O(\varepsilon)$*), and* $\mu_{L_1} > 0$*, such that*[a]

$$|(x(t), \xi(t))| \leq \beta_{L_1}(|(x_0, \xi_0)|, t - t_0) + \gamma_{L_1}(|u[t_0, t]|) + \tilde{\gamma}_\varepsilon(|u[t_0, t]|) \\ + \hat{\gamma}_\varepsilon(|\dot{u}[t_0, t]|) + \mu_{L_1}, \tag{4.17}$$

for all $\varepsilon \in (0, \tilde{\varepsilon}^*)$ *and for all* $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^m$*,* $u, \dot{u} \in \mathcal{L}_\infty$ *and* $t \geq t_0 \geq 0$*. Furthermore, the system (4.6) satisfies*

$$\int_{t_0}^{t} \alpha_V(|(x(s), \xi(s))|)ds \leq \overline{\alpha}_V(|(x_0, \xi_0)|) + \int_{t_0}^{t} \gamma_V(|u(s)|)ds + \varepsilon \int_{t_0}^{t} \tilde{\gamma}_V(|u(s)|)ds \\ + \varepsilon \int_{t_0}^{t} \hat{\gamma}_V(|\dot{u}(s)|)ds + \mu_V(t - t_0), \tag{4.18}$$

for all $\varepsilon \in (0, \tilde{\varepsilon}^*)$*,* $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^m$*, for all essentially bounded inputs with essentially bounded derivatives (i.e.* $u, \dot{u} \in \mathcal{L}_\infty$*) that belongs to* $\mathcal{L}_2$ *(i.e.* $u, \dot{u} \in \mathcal{L}_2$*), and for all* $t \geq t_0 \geq 0$*.*

The proof of Lemma 4.1 is presented in Appendix B.1. Lemma 4.1 implies that both slow and fast states are bounded. Now, we state that the fast states are ultimately bounded by a constant term that one can make arbitrarily small by reducing ε. We also show that the $\mathcal{L}_2$ upper bound is parametrized by ε, which is a desired property in this framework.

Corollary 4.1. *Consider the singularly perturbed system (4.6). If Assumptions 4.1 - 4.5 hold, there exists* $\beta_\xi(\cdot, \cdot) \in \mathcal{KL}$*, such that for any* $\tilde{\Delta} > 0$*,* $\tilde{\Delta}_{u_1} > 0$*,* $\tilde{\Delta}_{u_2} > 0$ *and* $\tilde{\mu} > 0$*, there exists* $\overline{\varepsilon}^* > 0$*, such that*

$$|\xi(t)| \leq \max\left\{\beta_\xi\left(|\xi_0|, \frac{t - t_0}{\varepsilon}\right), \tilde{\mu}\right\}, \tag{4.19}$$

for all $\varepsilon \in (0, \overline{\varepsilon}^*)$*,* $|(x_0, \xi_0)| \leq \tilde{\Delta}$*,* $||u||_\infty \leq \tilde{\Delta}_{u_1}$*,* $||\dot{u}||_\infty \leq \tilde{\Delta}_{u_2}$ *and* $t \geq t_0 \geq 0$*. Consequently, there exists a time* $\overline{T}^* > 0$ *such that*

$$|\xi(t)| \leq \tilde{\mu}, \tag{4.20}$$

for all $\varepsilon \in (0, \overline{\varepsilon}^*)$*,* $|(x_0, \xi_0)| \leq \tilde{\Delta}$*,* $||u||_\infty \leq \tilde{\Delta}_{u_1}$*,* $||\dot{u}||_\infty \leq \tilde{\Delta}_{u_2}$ *and* $t \geq \varepsilon\overline{T}^* + t_0 > 0$*. Furthermore, there exists* $\alpha_{W_c}(\cdot) \in \mathcal{K}_\infty$*, such that for the given* $\tilde{\Delta} > 0$*,* $\tilde{\Delta}_{u_1} > 0$*,* $\tilde{\Delta}_{u_2} > 0$

[a] In the sequel, $x_0 := x(t_0)$. The same applies for the other states.

and $\tilde{\mu} > 0$ *there exist* $\bar{\varepsilon}^*_{\mathcal{L}_2} > 0$*, such that*

$$\int_{t_0}^{t} \alpha_W(|\xi(s)|)ds \le \varepsilon \alpha_{W_c}(|\xi_0|) + \tilde{\mu}(t - t_0), \tag{4.21}$$

for all $\varepsilon \in (0, \bar{\varepsilon}^*_{\mathcal{L}_2})$*,* $|(x_0, \xi_0)| \le \tilde{\Delta}$*, for any input satisfying* $\|u\|_\infty \le \tilde{\Delta}_{u_1}$*,* $\|\dot{u}\|_\infty \le \tilde{\Delta}_{u_2}$*,* $|u|_{\mathcal{L}_2} \le \tilde{\Delta}_{u_1}$*,* $|\dot{u}|_{\mathcal{L}_2} \le \tilde{\Delta}_{u_2}$*, and for all* $t \ge t_0 \ge 0$*.*

The proof of Corollary 4.1 is presented in Appendix B.2. The proof relies on the analysis of the fast dynamics by considering the Lyapunov function for the boundary-layer system and Lemma 4.1. Note that the statement of the corollary implies that the ultimate bound in (4.19) and (4.20) can be made arbitrarily small. To do so, it is required to reduce ε, i.e., the magnitude of the ultimate bound determines the maximum value that the perturbation parameter can take. The properties given by Corollary 4.1 are exploited later in the proof of our main result.

4.4 Estimation error convergence result

We now study the robustness to singular perturbations of nonlinear observers of general dimension designed for the reduced system (4.3) and implemented on the system (4.6) by assuming that y and u are available. We follow the standard procedure on linear/nonlinear observer design for singularly perturbed systems; 1) we approximate the full system (4.3) by (4.3) and (4.9), 2) we then design an observer using the reduced (slow) system, and 3) we implement the observer synthesised for the reduced system on the original plant. We analyse the performance of the estimation error in the full system and prove that, under reasonable general conditions, the approach mentioned above leads to ISS and $\mathcal{L}_2$ stability properties for the error dynamics. Furthermore, we conclude useful SPA convergence results for the estimation error in the absence of measurement noise.

Here, we provide a general set of conditions to cover a large class of plant models and observers of general dimension. Since we assume that a nonlinear observer exists, our results are prescriptive. However, our conditions justify the use of a broader class of observers than those results in [68]. While authors in [68] deal with nonlinear systems where the slow part of the model satisfies a Lipschitz condition and the fast dynamics and the output of the plant are linear, here we consider a more general class of nonlinear plants. Moreover, whilst results in [68] only apply to a specific nonlinear

Luenberger-type observer that exhibits a linear error dynamics for the reduced system, our framework can cover a number of nonlinear observers including reduced-order [9], full-order [9, 26, 27, 43, 61, 68, 107–110], and higher-order observers [14].

We now assume that a nonlinear observer is designed for the reduced system (4.3). So, we consider the observer with the following general dynamics

$$\dot{\chi} = f_o(t, \chi, y_s, u), \tag{4.22a}$$

$$\hat{x} = h_o(t, \chi, u), \tag{4.22b}$$

where $\chi \in \mathbb{R}^q$ is the state of the observer, $\hat{x} \in \mathbb{R}^n$ is the output of the observer and an estimate of x (slow variable), y_s and u are the output and input of the nonlinear reduced system (4.3). In general, q is arbitrary and not necessarily equal to n so that (4.22) covers observers of general dimension. Existing results on nonlinear observer design for singularly perturbed systems only cover a Luenberger-type full-order observer [68].

Remark 4.2. *In the case of reduced-order observers, the observer dynamics are generally designed on the basis of an auxiliary subsystem. Then, for reduced-order observers, the output of the observer (4.22b) may depend on the output of the system (y_s). We allow this dependency only if y_s does not appear in the estimation error. We demonstrate in Chapter 5 that the reduced-order circle criterion observer in [9] satisfies this condition.*

Assumption 4.6. *The map $h_o(t, \chi, u) : [0, \infty) \times \mathbb{R}^q \times \mathbb{R}^r \to \mathbb{R}^n$ is a continuously differentiable function in all its arguments.*

Remark 4.3. *Assumption 4.6 implies that $h_o(t, \chi, u)$, $\partial h_o/\partial \chi$ and $\partial h_o/\partial u$ are continuous. Consider the variable $\hat{\chi} = [\chi, u]^T$, it follows from [Lemma 3.2, 2] that, for any $\Delta_1 > 0$ and $\Delta_2 > 0$, there exists $L_0 > 0$ so that when $|\partial h_o/\partial \hat{\chi}| \leq L_o$, for all χ and u such that $|\chi| \leq \Delta_1$ and $|u| \leq \Delta_2$, there exists $L > 0$ such that $|\partial h_o/\partial \chi| \leq L$. We have verified that Assumption 4.6 holds for reduced-order [9], full-order [9, 26, 27, 43, 61, 68, 107–110], and higher-order observers [14]. In Chapter 5, we present four of these observers.*

Define the estimation error as $e = \hat{x} - x$ so that the error dynamics for the observer synthesised for the reduced system (4.3) are given by

$$\dot{e} = f_e(t, x, \chi, e, H(t, x, u), y_s, u, \dot{u}, 0). \tag{4.23}$$

where

$$f_e(t, x, \chi, e, H(t, x, u), y_s, u, \dot{u}, 0) = \frac{\partial h_o}{\partial t} + \frac{\partial h_o}{\partial \chi} f_o(t, \chi, y_s, u) + \frac{\partial h_o}{\partial u}\dot{u}$$
$$- f_s(t, x, H(t, x, u), u, 0).$$

Remark 4.4. *For reduced-order observers relying on Remark 2, the error dynamics are defined by considering an auxiliary system used to construct the observer dynamics and the observer model itself. Later in Chapter 5, we illustrate this statement and Remark 4.2 via a reduced-order circle criterion observer presented in [9].*

Since the observer (4.22) is designed for the reduced order system (4.3), we require of an appropriate stability property for the error dynamics (4.23) to be able to analyse the performance of the observer when implemented on (4.6). We give generality to our framework by allowing to (4.23) to be input-to-state stable with respect to the measurement noise.

Assumption 4.7. *For the error dynamics in (4.23), there exists a continuously differentiable function* $V_3(t, e)$*, class-*$\mathcal{K}_\infty$ *functions* $\underline{\alpha}_{V_3}(\cdot)$, $\overline{\alpha}_{V_3}(\cdot)$, $\alpha_{V_3}(\cdot)$, $\gamma_{V_3}(\cdot)$*, and* $\zeta_2 > 0$, $\hat{\zeta}_2 > 0$*, such that for all* $(x, e) \in \mathbb{R}^n \times \mathbb{R}^n$, $u \in \mathbb{R}^r$, $t \geq 0$

$$\underline{\alpha}_{V_3}(|e|) \leq V_3(t, e, x, \chi) \leq \overline{\alpha}_{V_3}(|e|), \tag{4.24}$$

$$\frac{\partial V_3}{\partial t} + \frac{\partial V_3}{\partial e} f_e(t, x, \chi, e, H(t, x, u), y_s, u, \dot{u}, 0) + \frac{\partial V_3}{\partial x} f_s(t, x, H(t, x, u), u, 0)$$
$$+ \frac{\partial V_3}{\partial \chi} f_o(t, \chi, y_s, u) \leq -\zeta_2 \alpha_{V_3}^2(|e|) + \gamma_{V_3}(|w|), \tag{4.25}$$

$$\left|\frac{\partial V_3}{\partial e}\right| \leq \hat{\zeta}_2 \alpha_{V_3}(|e|). \tag{4.26}$$

The reduced-order observer presented in [9], the full-order observers [9, 26, 27, 43, 61, 68, 107–110], and higher-order observer [14] satisfy Assumption 4.7. Note that in the case of reduced-order observers $e \in \mathbb{R}^q$ where $q < n$. The Lyapunov condition (4.26) is common when one wants to use a Lyapunov function to prove robustness of a stability property, which is the case in this work. We verify this in a set of nonlinear observers presented in Chapter 5.

Remark 4.5. *It can be proven that Assumption 4.7 implies a boundedness of solutions property for the observer dynamics when* $q = n$*, i.e., the observer is of full-order. This can also be shown for reduced-order observers like the one in [Section 5.2, 9]. Note that* *the estimation error completely captures the behaviour of the observer state in full-order observers and some particular reduced-order observers. To show a boundedness property from Assumption 4.7, one can use* $V_3(t, \chi)$ *as a candidate Lyapunov function for the observer dynamics; then, by using some mild conditions on* f_s *and using (4.26) the result can be proven. For instance, we can show the result if* f_s *satisfies certain Lipschitz properties.*

In general, when $q \neq n$ (higher order and reduced order observers), it might be complicated or even impossible to show boundedness of solutions of the observer dynamics by just using Assumption 4.7. Hence, we need to assume that the observer has bounded solutions since we need such a property to prove a robustness result for observers of general dimension. This property is stated in the following assumption where y and u are seeing as input signals.

Assumption 4.8. *For the observer dynamics (4.22), there exist class-*$\mathcal{K}_\infty$ *functions* $\alpha_{o_1}(\cdot)$, $\alpha_{o_2}(\cdot)$, *and* $\alpha_{o_3}(\cdot)$, *such that for all* $\chi_0 \in \mathbb{R}^q$, $y, u \in \mathcal{L}_\infty$, $t \geq t_0 \geq 0$

$$|\chi(t)| \leq \alpha_{o_1}(|\chi_0|) + \alpha_{o_2}(\|y\|_\infty) + \alpha_{o_3}(\|u\|_\infty). \tag{4.27}$$

Moreover, if $u(t)$ *and* $y(t)$ *are essentially bounded input signals to the observer dynamics (4.22), there exist* $\alpha_{o_4}(\cdot), \alpha_{o_5}(\cdot), \alpha_{o_6}(\cdot), \alpha_{o_7}(\cdot) \in \mathcal{K}_\infty$, *such that*

$$\int_{t_0}^{t} \alpha_{o_4}(|\chi(\tau)|)d\tau \leq \alpha_{o_5}(|\chi_0|) + \int_{t_0}^{t} \alpha_{o_6}(|y(\tau)|)d\tau + \int_{t_0}^{t} \alpha_{o_7}(|u(\tau)|)d\tau. \tag{4.28}$$

To prove the robustness of the observer (4.22), its dynamics must have some kind of boundedness of solutions property when implemented on the original system. We use Assumption 4.8 to show that the observer states are ultimately bounded when we apply it to the original singularly perturbed system. Moreover, we prove that the observer states has a practical $\mathcal{L}_2$ stability property in the sense of [Property I3, 95].

Since we intend to estimate the slow variables of the singularly perturbed system (4.6), the observer synthesised based on the reduced order model (3.3) must be implemented on the plant (4.6). Then, the observer and error dynamics will be affected by the influence of the perturbation parameter ε as well as by the fast state $\xi \in \mathbb{R}^m$. Hence, we study the following systems

$$\dot{\chi} = f_o(t, \chi, y, u), \tag{4.29a}$$

$$\dot{e} = f_e(t, x, \chi, e, \xi + H(t, x, u), y, u, \dot{u}, \varepsilon), \tag{4.29b}$$

where

$$f_e(t, x, \chi, e, \xi + H(t, x, u), y, u, \dot{u}, \varepsilon) = \frac{\partial h_o}{\partial t} + \frac{\partial h_o}{\partial \chi} f_o(t, \chi, y, u) + \frac{\partial h_o}{\partial u}\dot{u} \\ - f_s(t, x, \xi + H(t, x, u), u, \varepsilon).$$

Note that the extended state (x, χ, e, ξ) represents the interconnection between the system (4.6), and the observer and error dynamics in (4.29). Hence, the robustness of the observer (4.22) is studied through the analysis of the estimation error performance in the full extended interconnected system given by

$$\dot{x} = f_s(t, x, \xi + H(t, x, u), u, \varepsilon), \tag{4.30a}$$

$$\dot{\chi} = f_o(t, \chi, y, u), \tag{4.30b}$$

$$\dot{e} = f_e(t, x, \chi, e, \xi + H(t, x, u), y, u, \dot{u}, \varepsilon), \tag{4.30c}$$

$$\varepsilon\dot{\xi} = f_f(t, x, \xi + H(t, x, u), u, \varepsilon) - \varepsilon\frac{\partial H}{\partial t} - \varepsilon\frac{\partial H}{\partial u}\dot{u} \\ - \varepsilon\frac{\partial H}{\partial x} f_s(t, x, \xi + H(t, x, u), u, \varepsilon), \tag{4.30d}$$

$$y = h(t, x, \xi + H(t, x, u), u, w, \varepsilon). \tag{4.30e}$$

Clearly, the observer dynamics are in a cascade with the state of the plant (x, ξ), while the error dynamics are in cascade with the extended state (x, χ, ξ). We exploit these properties to conclude our main result by using results in the previous section.

Assumption 4.9. *Consider the output of the system (4.30). There exists class-$\mathcal{K}_\infty$ functions $\alpha_y(\cdot)$, $\gamma_y(\cdot)$ and $\gamma_w(\cdot)$, such that, for any $\hat{\Delta} > 0$, $\hat{\Delta}_{u_1} > 0$ and $\hat{\Delta}_w > 0$, there exists ε_y such that*

$$|h(t, x, \xi + H(t, x, u), u, w, \varepsilon)| \le \alpha_y(|(x, \xi)|) + \gamma_y(|u|) + \gamma_w(|w|). \tag{4.31}$$

for all $\varepsilon \in (0, \varepsilon_y)$ and for all $|(x, \xi)| \le \hat{\Delta}$, $|u| \le \hat{\Delta}_{u_1}$, $|w| \le \hat{\Delta}_w$ and $t \ge 0$.

Assumption 4.9 is a mild assumption that allows us to show in Corollary 4.2 that the solutions of the observer are bounded when the observer is implemented on the original system. Observe that any continuous map h that is zero at zero satisfies our assumption, see [114]. We use Assumption 4.9 in the proof of the main result to bound terms related to the output of the observer.

Corollary 4.2. *Consider the observer dynamics (4.30b). If Assumptions 4.1 - 4.6, 4.8 and 4.9 hold, there exists a class-$\mathcal{K}_\infty$ function $\alpha_{c_1}(\cdot)$, such that for any $\overline{\Delta} > 0$, $\overline{\Delta}_{u_1} > 0$, $\overline{\Delta}_{u_2} > 0$, and $\overline{\Delta}_w \geq 0$ there exists $\hat{\varepsilon}^* > 0$ and $\Upsilon > 0$ such that*

$$|\chi(t)| \leq \alpha_{c_1}(|\chi_0|) + \Upsilon, \tag{4.32}$$

for all $\varepsilon \in (0, \hat{\varepsilon}^*)$ and for all $|(x_0, \xi_0, \chi_0)| \leq \overline{\Delta}$, $\|u\|_\infty \leq \overline{\Delta}_{u_1}$, $\|\dot{u}\|_\infty \leq \overline{\Delta}_{u_2}$, $\|w\|_\infty \leq \overline{\Delta}_w$ and $t \geq t_0 \geq 0$. Furthermore, there exist class-$\mathcal{K}_\infty$ functions $\alpha_{c_2}(\cdot)$, $\alpha_{c_3}(\cdot)$, such that for the given $\overline{\Delta} > 0$, $\overline{\Delta}_{u_1} > 0$, $\overline{\Delta}_{u_2} > 0$, and $\overline{\Delta}_w \geq 0$ there exists $\hat{\varepsilon}^* > 0$ and $\Upsilon_{\mathcal{L}_2} > 0$ such that

$$\int_{t_0}^{t} \alpha_{c_2}(|\chi(\tau)|)d\tau \leq \alpha_{c_3}(|\chi_0|) + \Upsilon_{\mathcal{L}_2}(t - t_0), \tag{4.33}$$

for all $\varepsilon \in (0, \hat{\varepsilon}^*)$, $|(x_0, \xi_0, \chi_0)| \leq \overline{\Delta}$, for any input satisfying $\|u\|_\infty \leq \overline{\Delta}_{u_1}$, $\|\dot{u}\|_\infty \leq \overline{\Delta}_{u_2}$, $|u|_{\mathcal{L}_2} \leq \overline{\Delta}_{u_1}$, $|\dot{u}|_{\mathcal{L}_2} \leq \overline{\Delta}_{u_2}$, for any $\|w\|_\infty \leq \Delta_w$, $|w|_{\mathcal{L}_2} \leq \Delta_w$ and for all $t \geq t_0 \geq 0$.

The proof of Corollary 4.2 is presented in Appendix B.3. Note that the above result implies that the states of the observer will remain bounded even under the influence of the fast variable $\xi \in \mathbb{R}^m$ and the singular perturbation parameter ε. We use this property to guarantee that those terms of the error dynamics related to $\chi \in \mathbb{R}^q$ remain bounded for all time.

Since we study the robustness of the stability property of the error dynamics when the observer (4.22) is implemented on the original system (4.6), we compute the derivative of $V_3(t, e, x, \chi)$ along the solutions of (4.30) in our main proof. This leads to interconnection terms which, in general, are of sign indefinite. Hence, we need conditions to bound those terms. Note that we have verified that several observers satisfy the following assumption. In fact, we show how this assumption hold on four different observers in Chapter 5.

Assumption 4.10. *Consider $\alpha_{V_1}(\cdot)$, $\alpha_W(\cdot)$ and $\alpha_{V_3}(\cdot)$ given in Assumptions 4.3, 4.4, and 4.7, respectively. Suppose there exist non-negative constants a_i and b_i $(i = 4, \ldots, 7)$, and class-$\mathcal{K}_\infty$ functions $\gamma_5(\cdot)$ and $\gamma_6(\cdot)$, so that the following conditions hold*

$$\begin{aligned}\left|\frac{\partial V_3}{\partial x}[f_s(t, x, \xi + H(t, x, u), u, \varepsilon) - f_s(t, x, H(t, x, u), u, 0)]\right| &\leq \varepsilon a_4 \alpha_{V_1}(|x|)\alpha_{V_3}(|e|) \\ &+ \varepsilon\gamma_5(|u|)\alpha_{V_3}(|e|) + b_4\alpha_{V_3}(|e|)\alpha_W(|\xi|), \end{aligned} \tag{4.34}$$

$$\left|\frac{\partial V_3}{\partial \chi}[f_o(t, \chi, y, u) - f_o(t, \chi, y_s, u)]\right| \leq \varepsilon a_5 \alpha_{V_1}(|x|)\alpha_{V_3}(|e|)$$

$$+ b_5\alpha_{V_3}(|e|)\alpha_W(|\xi|), \qquad (4.35)$$

$$\left|\frac{\partial V_3}{\partial e}[f_s(t,x,\xi+H(t,x,u),u,\varepsilon) - f_s(t,x,H(t,x,u),u,0)]\right| \le \varepsilon a_6\alpha_{V_1}(|x|)\alpha_{V_3}(|e|)$$

$$+\varepsilon\gamma_6(|u|)\alpha_{V_3}(|e|) + b_6\alpha_{V_3}(|e|)\alpha_W(|\xi|), \qquad (4.36)$$

$$\left|\frac{\partial V_3}{\partial e}\left[f_o(t,\chi,y,u) - f_o(t,\chi,y_s,u)\right]\right| \le \varepsilon a_7\alpha_{V_3}(|e|)\alpha_{V_1}(|x|)$$

$$+ b_7\alpha_{V_3}(|e|)\alpha_W(|\xi|). \qquad (4.37)$$

for all $(x,\xi,e,\chi) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^q$, $u \in \mathbb{R}^r$, $\dot{u} \in \mathbb{R}^r$ *and* $t \ge 0$.

Remark 4.6. *Assumption 4.10 can be relaxed to hold regionally or locally. Moreover, conditions in Assumption 4.10 can be relaxed to hold semi-globally with respect to the singular perturbation parameter* ε. *This means that for any positive constants* $\hat{\delta}_1 > 0$, $\hat{\delta}_2 > 0$, $\hat{\delta}_3 > 0$ *there exists* $\varepsilon^*_{a_{10}} > 0$ *such that (4.34) - (4.37) holds for all* $\varepsilon \in (0, \varepsilon^*_{a_{10}})$ *and for all* $|(x,\xi)| \le \hat{\delta}_1$, $|u| \le \hat{\delta}_2$, $|\dot{u}| \le \hat{\delta}_2$ *and* $t \ge 0$. *Our proofs are such that our results can be easily extended to cover these cases.*

Remark 4.7. *For reduced-order observers that satisfy conditions in Remarks 4.2 and 4.4, the notation on all of the above assumptions and definitions must be slightly modified. However, these modifications do not affect the essence of the assumptions, proofs and results.*

Our goal is to guarantee the convergence of the slow state estimate to the real value when the observer (4.22) synthesised for the reduced model (4.3) is applied on the full system (4.6). In the next section, we use results in Lemma 4.1 and Corollaries 4.1 and 4.2 to prove that the estimation error is semi-global practical ISS stable. We also prove a $\mathcal{L}_\infty \cap \mathcal{L}_2$ stability property for the error dynamics. Moreover, we show through a useful result that, in the absence of measurement noise, the error dynamics are SPA stable and that the ultimate bound for the error dynamics can be reduced by reducing ε.

4.4.1 Robustness Analysis

We first provide a useful result that states the error dynamics are ISS with respect to x, ξ, u and w. Moreover, we show the system (4.30) satisfies a $\mathcal{L}_\infty \cap \mathcal{L}_2$ stability property which is important in a broad range of applications when the measurement noise belongs to $\mathcal{L}_\infty \cap \mathcal{L}_2$. These results are the key ingredients that allow us to prove our main result in Theorem 4.1.

Lemma 4.2. *Consider the singularly perturbed system (4.30). If Assumptions 4.1 - 4.10 hold, there exists $\beta_e(\cdot,\cdot) \in \mathcal{KL}$, functions $\gamma_\xi(\cdot), \gamma_w(\cdot) \in \mathcal{K}_\infty$, and class-$\mathcal{K}_\infty$ functions $\gamma_{x,\varepsilon}(\cdot)$, $\gamma_{u,\varepsilon}(\cdot)$ parametrized by ε (their argument is of order $O(\varepsilon)$), such that for any $\Delta_L > 0$, $\Delta_{L_{u_1}} > 0$, $\Delta_{L_{u_2}} > 0$, and $\Delta_{L_w} > 0$, there exists $\varepsilon_L^* > 0$, such that*

$$|e(t)| \leq \beta_e(|e_0|, t - t_0) + \gamma_\xi(|\xi[t_0, t]|) + \gamma_{x,\varepsilon}(|x[t_0, t]|) + \gamma_{u,\varepsilon}(|u[t_0, t]|) + \gamma_w(|w[t_0, t]|), \tag{4.38}$$

for all $\varepsilon \in (0, \varepsilon_L^*)$ and for all $|(x_0, \xi_0, \chi_0, e_0)| \leq \Delta_L$, $\|u\|_\infty \leq \Delta_{L_{u_1}}$, $\|\dot{u}\|_\infty \leq \Delta_{L_{u_2}}$, $\|w\|_\infty \leq \Delta_{L_w}$ and $t \geq t_0 \geq 0$. Furthermore, there exists $k_1 > 0$ and $k_i \geq 0$ $(i = 2, 3)$ such that

$$\int_{t_0}^{t} \alpha_{V_3}^2(|e(\tau)|)d\tau \leq k_1\overline{\alpha}_{V_3}(|e_0|) + \varepsilon k_2 \int_{t_0}^{t} \alpha_{V_1}^2(|x(\tau)|)d\tau + \varepsilon k_1 \int_{t_0}^{t} \left[\gamma_5^2(|u(\tau)|) + \gamma_6^2(|u(\tau)|)\right] d\tau$$
$$+ k_3 \int_{t_0}^{t} \alpha_W^2(|\xi(\tau)|)d\tau + k_1 \int_{t_0}^{t} \gamma_{V_3}(|w(\tau)|)d\tau, \tag{4.39}$$

for all $\varepsilon \in (0, \varepsilon_L^*)$, $|(x_0, \xi_0, \chi_0, e_0)| \leq \Delta_L$, for any input satisfying $\|u\|_\infty \leq \Delta_{L_{u_1}}$, $\|\dot{u}\|_\infty \leq \Delta_{L_{u_2}}$, $\|u\|_{\mathcal{L}_2} \leq \Delta_{L_{u_1}}$, $\|\dot{u}\|_{\mathcal{L}_2} \leq \Delta_{L_{u_2}}$, for any $\|w\|_\infty \leq \Delta_{L_w}$, $\|w\|_{\mathcal{L}_2} \leq \Delta_{L_w}$ and $t \geq t_0 \geq 0$.

The proof of Lemma 4.2 is given in Appendix B.4. We now present the main result of this chapter in Theorem 4.1. This theorem states that the error dynamics exhibit a semi-global practical input-to-state stability property as well as a $\mathcal{L}_\infty \cap \mathcal{L}_2$ stability property when $w \in \mathcal{L}_\infty \cap \mathcal{L}_2$. Our proof focuses on the convergence of the estimation error while the other states in (4.30) are bounded. It is crucial for our proof to take into account that the error dynamics are in cascade with the original system and the observer dynamics. The next theorem summarises our main result.

Theorem 4.1. *Consider the singularly perturbed system (4.30). If Assumptions 4.1 - 4.10 hold, there exists $\beta_{T_1}(\cdot,\cdot) \in \mathcal{KL}$ and $\gamma_{T_1}(\cdot) \in \mathcal{K}_\infty$, such that for any $\Delta > 0$, $\Delta_{u_1} > 0$, $\Delta_{u_2} > 0$, $\Delta_w > 0$ and $\mu > 0$, there exists $\mu_{T_1} = \mu_{T_1}(\mu) > 0$ and $\varepsilon^* > 0$ such that*

$$|e(t)| \leq \beta_{T_1}\left(|(x_0, \xi_0, e_0)|, t - t_0\right) + \gamma_{T_1}\left(|w[t_0, t]|\right) + \mu_{T_1} + \mu, \tag{4.40}$$

for all $\varepsilon \in (0, \varepsilon^*)$, and for all $|(x_0, \xi_0, \chi_0, e_0)| \leq \Delta$, $\|u\|_\infty \leq \Delta_{u_1}$, $\|\dot{u}\|_\infty \leq \Delta_{u_2}$, $\|w\|_\infty \leq \Delta_w$, and $t \geq t_0 \geq 0$. Furthermore, there exists $\overline{\beta}_{T_1}(\cdot,\cdot) \in \mathcal{KL}$ and $\overline{\gamma}_{T_1}(\cdot)\mathcal{K}_\infty$, such that for the given $\Delta > 0$, $\Delta_{u_1} > 0$, $\Delta_{u_2} > 0$, $\Delta_w > 0$ and $\mu > 0$, there exists $T^* > 0$ and $\varepsilon^* > 0$ such that

$$|e(t)| \leq \overline{\beta}_{T_1}\left(|e_0|, t - t_0\right) + \overline{\gamma}_{T_1}\left(|w[t_0, t]|\right) + \mu, \tag{4.41}$$

for all $\varepsilon \in (0, \varepsilon^*)$, $|(x_0, \xi_0, \chi_0, e_0)| \leq \Delta$, $\|u\|_\infty \leq \Delta_{u_1}$, $\|\dot{u}\|_\infty \leq \Delta_{u_2}$, $\|w\|_\infty \leq \Delta_w$, *and* $t \geq \varepsilon T^* + t_0$. *In addition, there exists* $k_{T_1} > 0$ *and* $\alpha_{T_1}(\cdot), \overline{\alpha}_{T_1}(\cdot) \in \mathcal{K}_\infty$ *such that for the given* $\Delta > 0$, $\Delta_{u_1} > 0$, $\Delta_{u_2} > 0$, $\Delta_w > 0$ *and* $\mu > 0$, *there exists* $\mu_{\mathcal{L}_2} = \mu_{\mathcal{L}_2}(\mu) > 0$ *and* $\varepsilon^*_{\mathcal{L}_2} > 0$ *such that*

$$\int_{t_0}^{t} \alpha_{V_3}^2(|e(\tau)|)d\tau \leq \alpha_{T_1}(|(x_0, \xi_0, e_0)|) + \varepsilon\overline{\alpha}_{T_1}(|(x_0, \xi_0, e_0)|) \\ + k_{T_1}\int_{t_0}^{t} \gamma_{V_3}(|w(\tau)|)d\tau + \mu_{\mathcal{L}_2}(t - t_0) + \mu(t - t_0), \tag{4.42}$$

for all $\varepsilon \in (0, \varepsilon^*_{\mathcal{L}_2})$, $|(x_0, \xi_0, \chi_0, e_0)| \leq \Delta$, *for any input satisfying* $\|u\|_\infty \leq \Delta_{u_1}$, $\|\dot{u}\|_\infty \leq \Delta_{u_2}$, $|u|_{\mathcal{L}_2} \leq \Delta_{u_1}$, $|\dot{u}|_{\mathcal{L}_2} \leq \Delta_{u_2}$, *for any* $\|w\|_\infty \leq \Delta_w$, $|w|_{\mathcal{L}_2} \leq \Delta_w$ *and for all* $t \geq t_0 \geq 0$. *Furthermore, there exists* $k_{T_1} > 0$ *and* $\alpha_{T_1}(\cdot) \in \mathcal{K}_\infty$ *such that for the given* $\Delta > 0$, $\Delta_{u_1} > 0$, $\Delta_{u_2} > 0$, $\Delta_w > 0$ *and* $\mu > 0$, *there exists* $T^* > 0$ *and* $\varepsilon^*_{\mathcal{L}_2} > 0$ *such that*

$$\int_{t_0}^{t} \alpha_{V_3}^2(|e(\tau)|)d\tau \leq \alpha_{T_1}(|e_0|) + k_{T_1}\int_{t_0}^{t} \gamma_{V_3}(|w(\tau)|)d\tau + \mu(t - t_0). \tag{4.43}$$

for all $\varepsilon \in (0, \varepsilon^*_{\mathcal{L}_2})$, $|(x_0, \xi_0, \chi_0, e_0)| \leq \Delta$, *for any input satisfying* $\|u\|_\infty \leq \Delta_{u_1}$, $\|\dot{u}\|_\infty \leq \Delta_{u_2}$, $|u|_{\mathcal{L}_2} \leq \Delta_{u_1}$, $|\dot{u}|_{\mathcal{L}_2} \leq \Delta_{u_2}$, *for any* $\|w\|_\infty \leq \Delta_w$, $|w|_{\mathcal{L}_2} \leq \Delta_w$ *and for all* $t \geq \varepsilon T^* + t_0$.

The proof of Theorem 4.1 is given in Appendix B.5. Note that in the absence of measurement noise, stronger conclusions can be obtained as an immediate consequence of Theorem 4.1. The next corollary presents these sharper results.

Corollary 4.3. *Consider the singularly perturbed system (4.30). Let Assumptions 4.1 - 4.10 hold and assume that* $w(t) = 0$ *for all* $t \geq t_0 \geq 0$. *Then, there exists* $\beta_c(\cdot, \cdot) \in \mathcal{KL}$ *such that for any* $\Delta > 0$, $\Delta_{u_1} > 0$, $\Delta_{u_2} > 0$ *and* $\mu > 0$, *there exists* $T^* > 0$ *and* $\varepsilon^* > 0$ *such that*

$$|e(t)| \leq \beta_c(|e_0|, t - t_0) + \mu, \tag{4.44}$$

for all $\varepsilon \in (0, \varepsilon^*)$, $|(x_0, \xi_0, \chi_0, e_0)| \leq \Delta$, $\|u\|_\infty \leq \Delta_{u_1}$, $\|\dot{u}\|_\infty \leq \Delta_{u_2}$, *and* $t \geq \varepsilon T^* + t_0$. *Furthermore, there exists* $\alpha_c(\cdot) \in \mathcal{K}_\infty$ *such that for the given* $\Delta > 0$, $\Delta_{u_1} > 0$, $\Delta_{u_2} > 0$ *and* $\mu > 0$, *there exists* $T^* > 0$ *and* $\varepsilon^*_{\mathcal{L}_2} > 0$ *such that*

$$\int_{t_0}^{t} \alpha_{V_3}^2(|e(\tau)|)d\tau \leq \alpha_c(|e_0|) + \mu(t - t_0). \tag{4.45}$$

for all $\varepsilon \in (0, \varepsilon^*_{\mathcal{L}_2})$, $|(x_0, \xi_0, \chi_0, e_0)| \leq \Delta$, *for any input satisfying* $\|u\|_\infty \leq \Delta_{u_1}$, $\|\dot{u}\|_\infty \leq \Delta_{u_2}$,

$|u|_{\mathcal{L}_2} \leq \Delta_{u_1}$, $|\dot{u}|_{\mathcal{L}_2} \leq \Delta_{u_2}$, *and for all* $t \geq \varepsilon T^* + t_0$.

The proof of Corollary 4.3 follows directly from Theorem 4.1 so that we do not present a proof for this result. Observe that both Theorem 4.1 and Corollary 4.3 state detailed properties of the error dynamics. Moreover, these results ensure that any existing nonlinear observer satisfying our assumptions would perform properly when used to estimate the slow state of a singularly perturbed system. We now enunciate important remarks regarding the implications of our results.

Remark 4.8. *Corollary 4.3 implies a SPA stability property for the error dynamics in the absence of measurement noise. It is semi-global because the result holds for a given set of initial conditions and bounded inputs with bounded derivatives. It is practical in the perturbation parameter because one can make* μ *arbitrarily small by reducing* ε. *And it is asymptotical because of the class-*$\mathcal{KL}$ *function* $\beta_e(\cdot,\cdot)$. *These results (Theorem 4.1 and Corollary 4.3) imply important robustness properties for a large class of plants and observers. So, we can choose any existing observer, satisfying this framework, to estimate the slow states of a singularly perturbed system that satisfies the given assumptions.*

Remark 4.9. *The global and semi-global assumptions for the boundary layer and reduced systems, and for the observer and error dynamics can be relaxed. If all assumptions hold on appropriate bounded sets, the results hold in a given region defined by those sets. Moreover, our approach is such that local results can easily be stated if the assumptions are relaxed to hold locally.*

Remark 4.10. *If the fast dynamics (4.1b) do not depend on the input* u, *there is no need of any conditions on* $\dot{u}$. *Moreover, the results would not depend on* $\dot{u}$ *either.*

Remark 4.11. *Let* (A_2, R_2) *and* (A_4, R_4) *be the pairs representing the assumptions and results from Chapter 2 and Chapter 4, respectively. When dealing with full-order observers, it is straightforward to show that* $A_2 \implies A_4$ *and* $R_2 \implies R_4$. *Since* $A_4 \implies R_4$, *it follows that* $A_2 \implies R_4$. *Hence, we are able to state semi-global conclusions under strong (global) assumptions by using results from this chapter. However, we can only conclude strong (global) results under strong (global) assumptions when using results from Chapter 2 as* $A_4 \not\Longrightarrow A_2$.

4.5 Conclusions of the chapter

We have generated a new rigorous estimation framework for a general class of singularly perturbed systems in the standard form by considering nonlinear observers of general dimension, i.e. reduced-order, full-order and higher-order observers. We tackled the estimation problem of the slow variables of general nonlinear systems with two time-scales. We analysed the performance of nonlinear observers designed based on the slow part of the system while the fast dynamics are neglected. Therefore, we studied and proved the robustness of the observers to singular perturbations.

We provided a set of general conditions to guarantee an appropriate performance of the error dynamics when the observer synthesised for the reduced model is implemented on the full system. We delivered results that show robustness of observers with respect to singular perturbations and with respect to measurement noise. The main contribution of this chapter fills in a gap in the literature since we are unaware of such a general observer design framework for general nonlinear singularly perturbed systems. Moreover, the inclusion of the measurement noise gives significance to this work since this problem has not been considered before within the nonlinear singular perturbations framework.

Although the estimation of the slow states of a singularly perturbed system is usually done by neglecting the fast dynamics, there are no rigorous theoretical results that support such an approach. Hence, our results constitute a useful mathematical tool to justify the observer design for the slow states by just considering the reduced (slow) model. In summary, we have provided a general design framework to justify the use of observers synthesised for the slow system to estimate the slow states of two time-scale nonlinear plants.

Chapter 5

Applications of Semi-Global Results

In this chapter, we demonstrate that results in Chapter 4 cover existing results in [68]. Moreover, we present three different classes of plants, a full-order, a reduced-order and a higher-order nonlinear observers that are covered by the results in Chapter 4. We verify that the stated assumptions in Chapter 4 are satisfied such that we can demonstrate the applicability of our design framework from Chapter 4. We present simulation results to illustrate our theoretical findings.

5.1 Introduction

RESULTS in [68] apply to nonlinear singularly perturbed systems with linear fast dynamics, linear output and slow dynamics satisfying a Lipschitz condition. Their approach only allows to obtain conclusions for a specific Luenberger-type nonlinear observer. Therefore, this work generalises findings in [68] since we cover the results in there and we deal with a more general class of systems and estimators. Moreover, we have studied the robustness of the error dynamics with respect to measurement noise within the singular perturbations framework.

Our theoretical contributions in Chapter 4 apply to a large class of systems and nonlinear observers of general dimension. We have delivered a general observer design framework. Hence, we now demonstrate and illustrate the generality and usefulness of our results. We first show that our framework covers results in [68]. We then present three classes of singularly perturbed plants and nonlinear observers for which our results hold. In this chapter, we show that our results cover at least one full-order [27], one reduced-order [9] and one higher-order [14] observer. Moreover, we illustrate our results through numerical examples. Note that the classes of systems and observers considered in here are not covered by existing results in the literature. Although not presented here, our results cover the situation when the reduced (slow) system is such that nonlinear observers in [9,26,27,43,61,68,107–110] can be used to estimate the slow variables.

5.2 Luenberger-type nonlinear observer

An important work on robustness analysis for the observer design for the slow states of singularly perturbed systems is presented in [68]. Although the authors present useful results, their conclusions are restrictive to a particular class of systems and to a specific nonlinear Luenberger-type observer. In this section, we demonstrate that the design framework presented in Chapter 4 covers the results reported in [68].

We study the class of plants considered in [68] which is a subclass of the general class of systems (4.1). So, let us consider the class of nonlinear singularly perturbed systems described by the following model

$$\dot{x} = f(x, z), \tag{5.1a}$$

$$\varepsilon \dot{z} = M_1 x + M_2 z, \tag{5.1b}$$

$$y = C_1 x + C_2 z, \tag{5.1c}$$

where $x \in \mathbf{X} \subset \mathbb{R}^n$ is the slow state, $z \in \mathbf{Z} \subset \mathbb{R}^m$ is the fast state, $y \in \mathbb{R}^p$ is the output, $\varepsilon > 0$ is the singular perturbation parameter, and C_1, C_2, M_1, M_2 are matrices of appropriate dimensions. It is assumed that the sets $\mathbf{X}$ and $\mathbf{Z}$ are compact sets containing the origin so that $\mathbf{X} \times \mathbf{Z}$ is compact too. Furthermore, the authors of [68] assume that the map $f(x, z)$ is a real analytic vector function defined on the compact set $\mathbf{X} \times \mathbf{Z}$ and that $f(0, 0) = 0$.

Assumption 5.1. *The matrix M_2 in (5.1b) is Hurwitz.*

Assumption 5.1 is needed since the approximation of the original system through the reduced order and the boundary layer systems is only possible if the matrix M_2 is Hurwitz. We now demonstrate how the given assumptions in Chapter 4 are satisfied by this class of systems. Note that Assumption 4.1 states that the input and its derivative belong to $\mathcal{L}_\infty$. Since the class of systems in (5.1) does not consider inputs, it follows that Assumption 4.1 trivially holds.

In the following, we use the standard singular perturbations technique to obtain the reduced and boundary layer systems. We set $\varepsilon = 0$ such that the system is restricted to the slow manifold

$$M_1 x + M_2 z = 0. \tag{5.2}$$

Then, we have that $H(x) = -M_2^{-1} M_1 x$ is an isolated solution for (5.2) so that Assumption 4.2 holds. We now use $H(x)$ to construct the reduced system given by

$$\dot{x} = f(x, -M_2^{-1} M_1 x), \tag{5.3a}$$

$$y_s = Cx. \tag{5.3b}$$

where $C = C_1 - C_2 M_2^{-1} M_1$.

Assumption 5.2. *The reduced system (5.3) is locally exponentially stable.*

Remark 5.1. *Results in [68] hold under the assumption of local exponential stability of the reduced system (5.3). This assumption is used to ensure closeness of solutions on the infinite time interval of the solutions of (5.1) with respect to the solutions of the reduced system and those of the boundary layer system. Here, we apply the design framework from Chapter 4 relying on Remark 4.9.*

By virtue of Assumption 5.2, we have from [Theorem 4.14, 70] that there is a function $V_1(x)$ that satisfies the inequalities

$$c_1 |x|^2 \leq V_1(x) \leq c_2 |x|^2, \tag{5.4}$$

$$\frac{\partial V_1}{\partial x} f(x, -M_2^{-1} M_1 x) \leq -c_3 |x|^2, \tag{5.5}$$

$$\left| \frac{\partial V}{\partial x} \right| \leq c_4 |x|, \tag{5.6}$$

for some positive constants c_1, c_2, c_3 and c_4. Then, it follows that Assumption 4.3 holds.

We now perform a change of variables $z = \xi - M_2^{-1} M_1 x$ so that we obtain that the original system (5.1) in the new variables (x, ξ) is given by

$$\dot{x} = f(x, \xi - M_2^{-1} M_1 x), \tag{5.7a}$$

$$\varepsilon \dot{\xi} = M_2 \xi + \varepsilon (M_2^{-1} M_1) f(x, \xi - M_2^{-1} M_1 x), \tag{5.7b}$$

$$y = Cx + C_2 \xi. \tag{5.7c}$$

Then, by using the fast time-scale $\tau = t/\varepsilon$ and setting $\varepsilon = 0$, we have that the boundary layer system has the following dynamics

$$\frac{d\xi}{d\tau} = M_2\xi. \tag{5.8}$$

Since M_2 is Hurwitz (Assumption 5.1), we have from [Theorem 4.6, 70] that for any given positive definite symmetric matrix Q_ξ there exists a positive definite symmetric matrix P_ξ that satisfies the following Lyapunov equation

$$P_\xi M_2 + M_2^T P_\xi = -Q_\xi. \tag{5.9}$$

To check Assumption 4.4, consider $W(\xi) = \xi^T P_\xi \xi$ as a candidate Lyapunov function for (5.8). Then, it follows that

$$\frac{\partial W}{\partial \xi} M_2\xi \leq -\lambda_{\min}\{Q_\xi\}|\xi|^2, \tag{5.10}$$

which implies that the boundary layer system is asymptotically stable. Therefore, Assumption 4.4 is satisfied with $\underline{\alpha}_W(|\xi|) = \lambda_{\min}\{P_\xi\}|\xi|^2$ and $\overline{\alpha}_W(|\xi|) = \lambda_{\max}\{P_\xi\}|\xi|^2$, $\alpha_W(|\xi|) = |\xi|$ and $\zeta_3 = -\lambda_{\min}\{Q_2\}$.

We have that $f(x, z)$ is a real analytic vector function on a compact set $\mathbf{X} \times \mathbf{Z}$, then it is continuously differentiable. The compactness of $\mathbf{X} \times \mathbf{Z}$ implies that the convex hull $\mathrm{co}(\mathbf{X} \times \mathbf{Z})$ is compact. Then, by [Lemma 3.1, 70], the continuous differentiability of $f(x, z)$ and the fact that $\mathrm{co}(\mathbf{X} \times \mathbf{Z})$ is a convex compact set imply the function $f(x, z)$ is locally Lipschitz on $\mathbf{X} \times \mathbf{Z}$. Since $f(0, 0) = 0$, it follows from the Lipschitz property of $f(x, z)$ that $|f(x, \xi - M_2^{-1}M_1x)| \leq L_1|x| + L_2|\xi - M_2^{-1}M_1x|$ and $|f(x, \xi - M_2^{-1}M_1x) - f(x, -M_2^{-1}M_1x)| \leq L_3|\xi|$ hold for all $(x, \xi) \in \mathbf{X} \times \mathbf{Z}$ for some non-negative constants L_1, L_2, and L_3 [Lemma 3.2, 70]. Moreover, from Converse Theorem [Theorem 4.14, 70], we know the Lyapunov function $V_1(x)$ satisfies (5.6). Then, it follows that Assumption 4.5 is satisfied with $a_1 = 0, \gamma_1(\cdot) = 0$, $b_1 = cL_3$, $a_2 = 0$, $b_2 = 0$, $\gamma_2(\cdot) = 0$, $a_3 = 2L_2|P_\xi||M_2^{-1}M_1|$, $b_3 = 2(L_1 + L_2|M_2^{-1}M_1|)|P_\xi||M_2^{-1}M_1|$, $\gamma_3(\cdot) = 0$ and $\gamma_4(\cdot) = 0$.

5.2.1 Observer design

We now consider the Luenberger-type nonlinear observer introduced in [69] with the following dynamics

$$\dot{\hat{x}} = f(\hat{x}) + L(\hat{x})(y - \hat{y}), \tag{5.11}$$

where $\hat{x} \in \mathbb{R}^n$ is the observer state and an estimate of $x \in \mathbb{R}^n$, and $\hat{y} = C\hat{x}$ is the estimated output. Since the output of the observer is the state by itself, it follows that Assumption 4.6 holds. The state-dependent gain $L(\hat{x})$ is defined as

$$L(\hat{x}) = \left[\frac{\partial T}{\partial \hat{x}}\right]^{-1} B, \tag{5.12}$$

where $T(\hat{x})$ is a solution to a system of partial differential equations given by

$$\frac{\partial T}{\partial \hat{x}} f(\hat{x}) = AT(\hat{x}) + BC\hat{x}, \tag{5.13}$$

with A and B being matrices of appropriate dimensions, and A being Hurwitz. It is shown in [68], that a coordinate transformation given by $w = T(x)$ leads to linear error dynamics when we define the estimation error as $e = w - \hat{w}$. It can be shown that the observation error for the reduced system (5.3) has the following linear dynamics

$$\dot{e} = Ae. \tag{5.14}$$

Since A is Hurwitz, it follows from [Theorem 4.6, 70] that for any given positive definite symmetric matrix Q_e there exists a positive definite symmetric matrix P_e that satisfies the following Lyapunov equation

$$P_e A + A^T P_e = -Q_e, \tag{5.15}$$

so that the quadratic function $V_2(e) = e^T P_e e$ is a Lyapunov function candidate for (5.14). Then, the derivative of $V_2(e)$ along the trajectories of the linear system (5.14) is bounded as follows

$$\dot{V}_2(e)\big|_{5.14} \leq -\lambda_{\min}\{Q_e\}|e|^2. \tag{5.16}$$

It follows that Assumption 4.7 holds with $\underline{\alpha}_{V_3}(|e|) = \lambda_{\min}\{P_e\}|e|^2$, $\overline{\alpha}_{V_3}(|e|) = \lambda_{\max}\{P_e\}|e|^2$, $\zeta_2 = \lambda_{\min}\{Q_e\}$, $\alpha_{V_3}(|e|) = |e|$, and $\hat{\zeta}_2 = 2|P_e|$. Observe that the observer (5.11) is of full-order such that Assumption 4.8 holds relying on Remark 4.5. Moreover, since the output of the system (5.1) is linear and does not depend on the perturbation parameter, it follows that Assumption 4.9 trivially holds.

When the observer (5.11) is implemented on the full system (5.1), the error dynamics become nonlinear and are given by

$$\dot{e} = Ae - BC_2\xi + \frac{\partial T}{\partial x}\left[\, f(x, \xi - M_2^{-1}M_1x) - f(x, -M_2^{-1}M_1x)\right]. \tag{5.17}$$

Since our results hold when our assumptions are satisfied in appropriate bounded sets, we take advantage of some of the features of $f(x,z)$ and $T(x)$ to check Assumption 4.10. In the compact set **X**, the analyticity of the map $T(x)$ implies $|\partial T/\partial x| \leq L_4$ for all $x \in \mathbf{X}$ with $L_4 > 0$. As pointed out above, we have that $|f(x, \xi - M_2^{-1}M_1x) - f(x, -M_2^{-1}M_1x)| \leq L_3|\xi|$ holds for all $(x, \xi) \in \mathbf{X} \times \mathbf{Z}$ with $L_3 > 0$. So, we have that the norm of the difference between (5.14) and (5.17) is bounded as follows

$$\left| Ae - BC_2\xi + \frac{\partial T}{\partial x}\left[\, f(x, \xi - M_2^{-1}M_1x) \; - f(x, -M_2^{-1}M_1x)\right] - Ae\right| \leq$$
$$|BC_2||\xi| + L_3L_4|\xi|. \tag{5.18}$$

Then, it follows that Assumption 4.10 holds with $b_4 = 2|P_e|(|BC_2| + L_3L_4)$ and with all of the rest constants and functions equal to zero. As Assumptions 4.1 - 4.10 are satisfied, we conclude that Theorem 4.1 holds. We summarise the above results in the following corollary which is an immediate consequence of Theorem 4.1.

Corollary 5.1. *Consider the singularly perturbed plant (5.1), the nonlinear Luenberger-type observer (5.11) and the error dynamics (5.17). If Assumptions 5.1 and 5.2 hold, there exists a positive definite matrix* P_e *and a constant* $\mathbf{c} > 0$ *such that for any* $\Delta > 0$ *and* $\mu > 0$*, there exists* $\varepsilon^* > 0$ *such that*

$$|e(t)| \leq \sqrt{\frac{\lambda_{max}(P_e)}{\lambda_{min}(P_e)}}|e_0| \exp\left(-\frac{\mathbf{c}}{2\lambda_{max}(P_e)}(t - t_0)\right) + \mu, \tag{5.19}$$

for all $\varepsilon \in (0, \varepsilon^*)$, $|(x_0, \xi_0, e_0)| \leq \Delta$ *and* $t \geq t_0$.

5.3 Robust circle-criterion observer

In this section, we consider a class of singularly perturbed plants where the reduced (slow) model takes the form in which results from [27] can be applied to design a full-order observer. This class of plants is covered by the general model (4.1). Note that this class of systems and the observer are not covered by results in [68]. Moreover, this class of plants is more general than those considered in Chapter 3. Consider the class of plants with the following nonlinear singularly perturbed form

$$\dot{x} = Ax + Bz + G\gamma(F_1x + F_2z) + \sigma(y, u), \tag{5.20a}$$

$$\varepsilon\dot{z} = M_1x + M_2z, \tag{5.20b}$$

$$y = C_1x + C_2z + Dw, \tag{5.20c}$$

where the state vector $x \in \mathbb{R}^n$ corresponds to the slow state, $z \in \mathbb{R}^m$ is the fast state, $y \in \mathbb{R}^p$ is the measured output variable, $u \in \mathbb{R}^r$ is the control input, w is the measurement noise, $\varepsilon > 0$ is the singular perturbation parameter of the process, $\gamma(\cdot) = [\gamma_1(\cdot), \ldots, \gamma_{n_\gamma}(\cdot)]^T$ is a nondecreasing locally Lipschitz function, and A, B, F_1, F_2, G, C_1, C_2, D, M_1 and M_2 are matrices of appropriate dimensions. We require a linear dynamics in (5.20b) for two reasons: 1) it is easier to compute the slow manifold, and 2) with a linear fast dynamics we end up with a reduced model that exhibits a structure for which we can design the circle criterion observer in [27].

Assumption 5.3. *The matrix* M_2 *in (5.20b) is Hurwitz.*

Assumption 5.4. *The solutions of the system belong to a compact set. Moreover, the functions* $\gamma(\cdot)$ *and* $\sigma(\cdot,\cdot)$ *are locally Lipschitz, and* $\gamma(\cdot)$ *satisfies [Assumption 1, 27].*

Assumption 5.4 over $\sigma(\cdot,\cdot)$ is required to prevent the solutions of x from escaping to infinity in a finite time [9]. Note that this example satisfies our results relaying in Remark 4.9 since we consider a local Lipschitz condition. From Assumption 5.4, we know that for any i-entry (γ_i) of the vector γ, there exists a time-varying gain $\delta_i(t)$ taking values in the interval $[0, L_i]$ such that

$$\gamma_i(a_i) - \gamma_i(b_i) \leq \delta_i(t)(a_i - b_i), \; \forall \, a_i, b_i \in \mathbb{R}, \tag{5.21}$$

where L_i is a Lipschitz constant for γ_i. This property must hold in order to implement the circle criterion observer introduced in [27]. We now check our assumptions for the class of systems represented by (5.20). Note that Assumption 4.4 requires $u, \dot{u}, w \in \mathcal{L}_\infty$. It is observed that no condition is needed for $\dot{u}$ because the fast dynamics do not depend on u.

Assumption 5.5. *The input* u *and the measurement noise* w *are essentially bounded signals, i.e.* $u, w \in \mathcal{L}_\infty$.

It follows from the above assumption that Assumption 4.1 trivially holds. To obtain the lower dimensional systems, we set $\varepsilon = 0$ such that the system is restricted to the slow manifold

$$M_1x + M_2z = 0. \tag{5.22}$$

Then, it follows that, $H(x) = -M_2^{-1}M_1x$, is an isolated solution of (5.22). Then, Assumption 4.2 holds with $H(x) = -M_2^{-1}M_1x$ which always exists by virtue of Assumption 5.3. By using $H(x)$, we have that the reduced system is given by

$$\dot{x} = A_0x + G\gamma(\bar{F}x) + \sigma(y_s, u), \tag{5.23a}$$

$$y_s = Cx + Dw, \tag{5.23b}$$

where $A_0 = A - BM_2^{-1}M_1$, $C = C_1 - C_2M_2^{-1}M_1$, and $\bar{F} = F_1 - F_2M_2^{-1}M_1$. Note that it is assumed that the pair (A_0, C) is detectable. Therefore, to allow more generality for the matrix A_0, we need to assume that the reduced system (5.23) is input-to-state practically stable (ISpS), such that there exists a Lyapunov ISpS function that satisfies Assumption 4.3. This is required since there is no need for A_0 to be Hurwitz.

Remark 5.2. *If the matrix A_0 is Hurwitz, then Assumption 4.3 holds with $V_1(x)$ being a quadratic Lyapunov function. Moreover, it is straightforward to find the functions and constants for which Assumption 4.5 is satisfied.*

We now define the change of variables $z = \xi - M_2^{-1}M_1x$. Then, the original system (5.20) in the (x, ξ) variables is given by

$$\dot{x} = Ax + G\gamma(F_1x + F_2(\xi - M_2^{-1}M_1x)) + \sigma(y, u) + B(\xi - M_2^{-1}M_1x), \tag{5.24a}$$

$$\begin{aligned} \varepsilon\dot{\xi} = M_2\xi + \varepsilon(M_2^{-1}M_1)[&Ax + G\gamma(F_1x + F_2(\xi - M_2^{-1}M_1x)) + \sigma(y, u) \\ &+ B(\xi - M_2^{-1}M_1x)], \end{aligned} \tag{5.24b}$$

$$y = Cx + C_2\xi + Dw. \tag{5.24c}$$

By expressing (5.24) in the fast time-scale $\tau = t/\varepsilon$, we have that the boundary layer system at $\varepsilon = 0$ is given by

$$\frac{d\xi}{d\tau} = M_2\xi. \tag{5.25}$$

Since M_2 is Hurwitz, we have from [Theorem 4.6, 70] that for any given positive definite symmetric matrix Q_ξ there exists a positive definite symmetric matrix P_ξ that satisfies the following Lyapunov equation

$$P_\xi M_2 + M_2^T P_\xi = -Q_\xi. \tag{5.26}$$

To check Assumption 4.4, consider $W(\xi) = \xi^T P_\xi \xi$ as a candidate Lyapunov function for (5.25). It follows that

$$\frac{\partial W}{\partial \xi} M_2 \xi \leq -\lambda_{\min}\{Q_\xi\}|\xi|^2. \tag{5.27}$$

Therefore, Assumption 4.4 holds with $\underline{\alpha}_W(|\xi|) = \lambda_{\min}\{P_\xi\}|\xi|^2$ and $\overline{\alpha}_W(|\xi|) = \lambda_{\max}\{P_\xi\}|\xi|^2$ as the lower and upper bounds for $W(\xi)$ respectively, and with $\zeta_3 = \lambda_{\min}\{Q_\xi\}$ and $\alpha_W(|\xi|) = |\xi|$ as the terms satisfying (5.32). Due to the generality of the matrix A_0, we need to assume that the full system (5.24) satisfies the interconnection conditions in Assumption 4.5.

5.3.1 Observer design

We now consider the circle criterion observer proposed in [27] with the following dynamics

$$\dot{\hat{x}} = A_0\hat{x} + L(C\hat{x} - y) + G\gamma(\overline{F}\hat{x} + K(C\hat{x} - y)) + \sigma(y, u), \tag{5.28}$$

where $\hat{x} \in \mathbb{R}^n$ is the observer's state and an estimate of the state, K and L are gain matrices of appropriate dimensions which must be designed. By following the approach described in this manuscript, the observer (5.28) is designed for the reduced system (5.23), and then implemented on the full singularly perturbed plant (5.25). Since we are dealing with a full order observer, it follows that Assumption 4.6 trivially holds because the output of the observer is a linear map in which the transformation matrix is the identity matrix. We now define the estimation error as $e := x - \hat{x}$. It follows that the error dynamics are given by

$$\dot{e} = (A_0 + LC)e - LDw + G[\gamma(\overline{F}x) - \gamma(\overline{F}(x - e) - KCe - KDw)]. \tag{5.29}$$

To check Assumption 4.7, we consider the Lyapunov function $V_3(e) = e^T P_3 e$, where $P_3 = P_3^T > 0$. The matrix P_3 is obtained by solving the following LMI from [27]

$$\begin{bmatrix} (A_0 + LC)^T P_3 + P_3(A_0 + LC) + \hat{\nu} I & P_3 G + (F + KC)^T \Lambda & -P_3 LD \\ G^T P_3 + \Lambda(F + KC) & -2\Lambda\left(\frac{1}{L_1}, \cdots, \frac{1}{L_{n_\gamma}}\right) & -\Lambda KD \\ -P_3 LD & -\Lambda KD & \mu_w I \end{bmatrix} \leq 0, \tag{5.30}$$

where $\Lambda > 0$ is a diagonal matrix and an observer design parameter, $\mu_w > 0$ is a scalar constant and $\hat{\nu} > 0$ is also an observer design parameter. When the LMI in (5.30) is satisfied, it follows from [27] that

$$\frac{\partial V_3}{\partial e} f_e(x, e) \leq -\hat{\nu}|e|^2 + \mu_w |w|^2, \tag{5.31}$$

with $f_e(x, e) = (A_0 + LC)e - LDw + G[\gamma(\overline{F}x) - \gamma(\overline{F}(x - e) - KCe - KDw)]$. Then, Assumption 4.7 holds with $\underline{\alpha}_{V_3}(|e|) = \lambda_{\min}\{P_3\}|e|^2$ and $\overline{\alpha}_{V_3}(e) = \lambda_{\max}\{P_3\}|e|^2$ being the lower and upper bounds for $V_3(|e|)$ respectively, with $\zeta_2 = \hat{\nu}$, $\alpha_{V_3}(|e|) = |e|$ and $\gamma_{V_3}(|w|) = \mu_w |w|^2$ being the elements that satisfy the bound in (5.31), and with $\hat{\zeta}_2 = 2|P_3|$ being the constant that multiplies $\alpha_{V_3}(\cdot)$ to bound the norm of the gradient of $V_3(e)$ with respect to e.

It follows from Remark 4.5 that Assumption 4.8 holds when using the Lyapunov function $V_3(\hat{x})$ as a Lyapunov function for the observer dynamics (5.28). Moreover, Assumption 4.9 holds globally since the output does not depend on the perturbation parameter. Then, Assumption 4.9 is satisfied with $\alpha_y(|(x, \xi)|) = 2\max\{|C|, |C_2|\}|(x, \xi)|$ and $\gamma_w(|w|) = |D||w|$. Note that in the case of full order observers we do not need Assumptions 4.6, 4.8 and 4.9, and Corollary 4.2 since the error dynamics completely capture the performance of the observer states.

To verify Assumption 4.10, we need to obtain the error dynamics when the observer (5.28) is implemented on the full system (5.25). Then, by considering the full system we obtain that the error dynamics are given as follows

$$\begin{aligned} \dot{e} &= (A_0 + LC)e + G\gamma([\overline{F}, F_2][x, \xi]^T) + B\xi + LC_2\xi - LDw \\ &\quad - G\gamma(\overline{F}(x - e) - K(Ce + C_2\xi) - KDw). \end{aligned} \tag{5.32}$$

By considering Assumption 5.4 and equations (5.29) and (5.32), we have that Assumption 4.10 holds with $b_6 = 2(|P_1||B + LC_2| + L_0|P_1 G||F_2| + L_0|P_1 G||KC_2|)$, while the rest of the constants and functions are zero. L_0 is the Lipschitz constant on the compact set where the solutions belong to. We have checked that Assumption 4.1 - 4.10 hold for plants in the form of (5.20) and the circle criterion observer (5.28). Therefore, we conclude that all our results holds relaying on Remark 4.9. We summarise this section in the following corollary which is an immediate consequence of Theorem 4.1.

Corollary 5.2. *Consider the singularly perturbed plant (5.20), the circle criterion observer (5.28) and the error dynamics (5.32). If Assumptions 4.3, 4.5 and 5.3 - 5.5 hold, there exists a positive definite matrix* P_3*, constants* $\hat{\nu} > 0$ *and* $\mu_w > 0$*, such that for any* $\Delta > 0$*,* $\Delta_{u_1} > 0$*,* $\Delta_{u_2} > 0$*,* $\Delta_w > 0$ *and* $\mu > 0$*, there exists* $T^* > 0$ *and* $\varepsilon^* > 0$ *such that*

$$|e(t)| \leq \sqrt{\frac{\lambda_{\max}(P_3)}{\lambda_{\min}(P_3)}}|e_0| \exp\left(-\frac{\hat{\nu}}{2\lambda_{\max}(P_3)}t\right) + \sqrt{\frac{8\lambda_{\max}(P_3)}{\hat{\nu}\lambda_{\min}(P_3)}\mu_w \, \|w\,[t_0, t]\|^2} + \mu, \tag{5.33}$$

for all $\varepsilon \in (0, \varepsilon^*)$*,* $|(x_0, \xi_0, e_0)| \leq \Delta$*,* $\|u\|_\infty \leq \Delta_{u_1}$*,* $\|\dot{u}\|_\infty \leq \Delta_{u_2}$*,* $\|w\|_\infty \leq \Delta_w$*, and* $t \geq \varepsilon T^* + t_0$*. Furthermore, for the given* $\Delta > 0$*,* $\Delta_{u_1} > 0$*,* $\Delta_{u_2} > 0$*,* $\Delta_w > 0$ *and* $\mu > 0$*, there exists* $T^* > 0$ *and* $\varepsilon^*_{\mathcal{L}_2} > 0$ *such that*

$$\int_{t_0}^{t} |e(\tau)|^2 d\tau \leq \frac{2\lambda_{\max}(P_3)}{\hat{\nu}}|e_0| + \frac{2\mu_w}{\hat{\nu}}\int_{t_0}^{t} |w(\tau)|^2 d\tau + \mu(t - t_0). \tag{5.34}$$

for all $\varepsilon \in (0, \varepsilon^*_{\mathcal{L}_2})$*,* $|(x_0, \xi_0, e_0)| \leq \Delta$*, for any input satisfying* $\|u\|_\infty \leq \Delta_{u_1}$*,* $\|\dot{u}\|_\infty \leq \Delta_{u_2}$*,* $|u|_{\mathcal{L}_2} \leq \Delta_{u_1}$*,* $|\dot{u}|_{\mathcal{L}_2} \leq \Delta_{u_2}$*, for any* $\|w\|_\infty \leq \Delta_w$*,* $|w|_{\mathcal{L}_2} \leq \Delta_w$ *and for all* $t \geq \varepsilon T^* + t_0$*.*

5.3.2 Simulation results

Suspension system with nonlinear hardening spring

In this section, we present simulations results to illustrate the applicability of our results. We consider the quarter-car model of automotive suspension in [Chapter 11, 70] with nonlinear hardening spring between the car body and the tire. We assume that there is no disturbance to the system and define $\dot{\mathbf{x}} = d\mathbf{x}/dt_r$ with $t_r = t\sqrt{k_s/m_s}$. Then, the model of the system can be expressed in the following standard singularly perturbed form

$$\dot{x}_1 = x_2 - z_2, \tag{5.35a}$$

$$\dot{x}_2 = -x_1 - x_1^3 - \beta(x_2 - z_2) + u, \tag{5.35b}$$

$$\varepsilon\dot{z}_1 = z_2, \tag{5.35c}$$

$$\varepsilon\dot{z}_2 = \alpha x_1 + \alpha x_1^3 - \alpha\beta(z_2 - x_2) - z_1 - \alpha u, \tag{5.35d}$$

with $\varepsilon = \sqrt{\frac{k_s m_u}{k_t m_s}}$, $\alpha = \sqrt{\frac{k_s m_s}{k_t m_u}}$, $\beta = \frac{b_s}{\sqrt{k_s m_s}}$, $u = \frac{F}{k_s \ell}$, where m_s and m_u are the car body and tire masses, k_s and k_t are the spring constants of the strut and tire, ℓ is a constant distance, which is used to normalize variables, b_s is the shock absorber constant (damping term), and F is a bounded force generated by a force generator that acts as an active element. Note that the reduced system system is given by

$$\dot{x}_1 = x_2, \tag{5.36a}$$

$$\dot{x}_2 = -x_1 - x_1^3 - \beta x_2 + u. \tag{5.36b}$$

While the boundary layer system in the fast time scale $\tau - t/\varepsilon$, is defined by

$$\frac{d\xi_1}{d\tau} = \xi_2, \tag{5.37a}$$

$$\frac{d\xi_2}{d\tau} = -\alpha\beta\xi_2 - \xi_1. \tag{5.37b}$$

We assume that an accelerometer, located on the car body, is used to measure the vertical velocity of the system; such a sensor provides an output in the form of

$$y = x_2 + w(t), \tag{5.38}$$

where $w(t) = 0.05\sin(0.3t)$ is the measurement noise. Note that the reduced system (5.36) with the output (5.38) can be written in the form of (5.23) with $\sigma(y, u) = [y, u - \beta y]^T$, and $A = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$, $G = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $D = 1$. Therefore, it follows that a circle-criterion observer (5.28) can be used to estimate the slow states of (5.35). Hence, the estimation error is expected to converge to a region around the origin as highlighted in Corollary 5.2. The aforementioned region is critically related to ε and to the bound of the measurement noise.

To perform simulations, we consider the following parameters: $k_s = 500[N - m]$, $k_t = 6[KN - m]$, $m_s = 200[Kg]$, $m_u = 20[Kg]$, $b_s = 35[N - s/m]$. For these values, we have that $\alpha = 0.913$ and $\beta = 0.111$. Note that, for the given parameters, the perturbation parameter for the system is $\varepsilon = 0.0913$. By following the design procedure described above, we obtain the following gain matrices for the circle-criterion observer $K = 2.548$, $L = \begin{bmatrix} 1.503 \\ -1.281 \end{bmatrix}$. The performance of the Circle Criterion Observer designed for the slow system (5.36) and implemented on the full system (5.35) is presented in Figure 5.1 for different values of ε. It is observed that the estimation error performs as expected, i.e. it converges to a small offset around the origin. Even though the circle criterion observer has an exponential convergence rate for the reduced system, its performance on the original system is affected by the perturbation parameter and the fast part of the state as stated in our main result and illustrated in Figure 5.1.

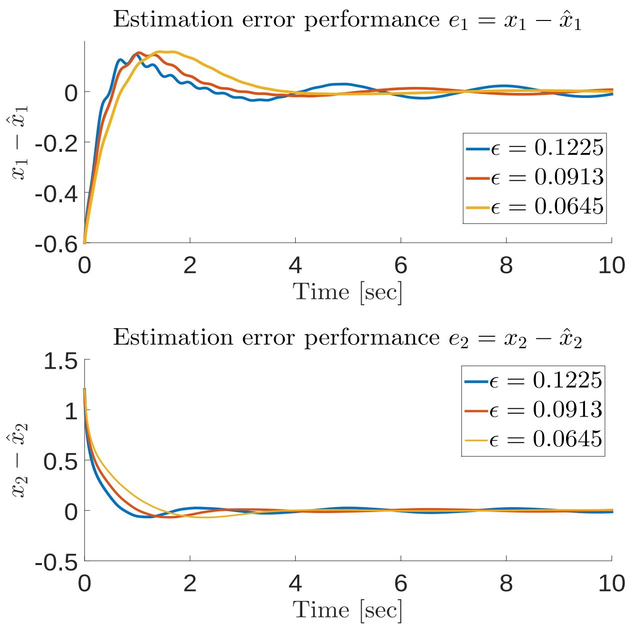

Figure 5.1: Estimation error performance for the estimates of x_1 and x_2 through the Circle Criterion Observer.

Single-link flexible joint robot

We now illustrate our findings in a second example. We present simulation results for the single-link flexible joint robot showed in Figure 5.2. This nonlinear system and its parameters were taken from [43]. The motor and the link position and velocities of the flexible joint robot are denoted by θ_m, ω_m, θ_l and ω_l, respectively. We assume the robot is instrumented with a sensor that measures the motor position and that has linear fast dynamics. Then, the model of the full system is given by

$$\dot{\theta}_m = \omega_m, \tag{5.39a}$$

$$\dot{\omega}_m = \frac{\tau}{J_m} - \frac{B}{J_m}\omega_m + \frac{K_\tau}{J_m}u, \tag{5.39b}$$

$$\dot{\theta}_l = \omega_l, \tag{5.39c}$$

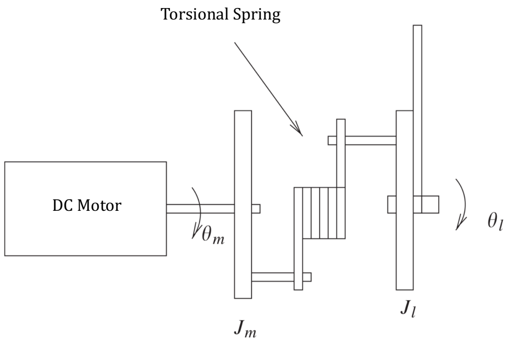

Figure 5.2: A flexible joint robot link with a stiffening torsional spring in the flexible joint.

$$\dot{\omega}_l = -\frac{\tau}{J_l} - \frac{mgh}{J_l}\sin(\theta_l), \tag{5.39d}$$

$$\varepsilon\dot{z} = \theta_m - z, \tag{5.39e}$$

$$y = z, \tag{5.39f}$$

where J_m is the inertia of the DC motor, J_l is the inertia of the link, $2h$ and m represent the length and mass of the link, B is the viscous friction, and K_τ is the amplifier gain. Since we consider a stiffening torsional spring in the flexible joint, the torque due to this spring is given by

$$\tau = \kappa_1(\theta_l - \theta_m) + \kappa_2(\theta_l - \theta_m)^3. \tag{5.40}$$

Observe that the model (5.39) agrees with the general plant in (5.20) so that we can estimate the slow variables by using an observer synthesised for the reduced model if the singular perturbation parameter is small enough. For this example, the singular perturbation parameter can be treated as a tunable parameter since it represents the time-constant of the sensor. We design a circle criterion observer and perform simulations by using the parameters given in Table 5.1.

The simulation results are shown in Figure 5.3. The performance of the slow state estimation error is related to the magnitude of ε. When the sensor dynamics are fast enough (small ε), such fast dynamics have no significant impact on the convergence of the estimation error. Observe that the estimation error does converges to neighbourhood around zero.

Table 5.1: System parameters for single-link robot arm

Inertia of the motor $(J_m[kgm^2])$	3.7×10^{-3}
Inertia of the link $(J_m[kgm^2])$	9.3×10^{-3}
Length of the link $(2h[m])$	3.1×10^{-1}
Mass of the link $(m[kg])$	0.21
Viscous friction $(B[m])$	$4.6\times^{-2}$
Amplifier gain $(K_\tau[NmV^{-1}])$	8×10^{-2}
Constant κ_1	1.8×10^{-1}
Constant κ_2	1

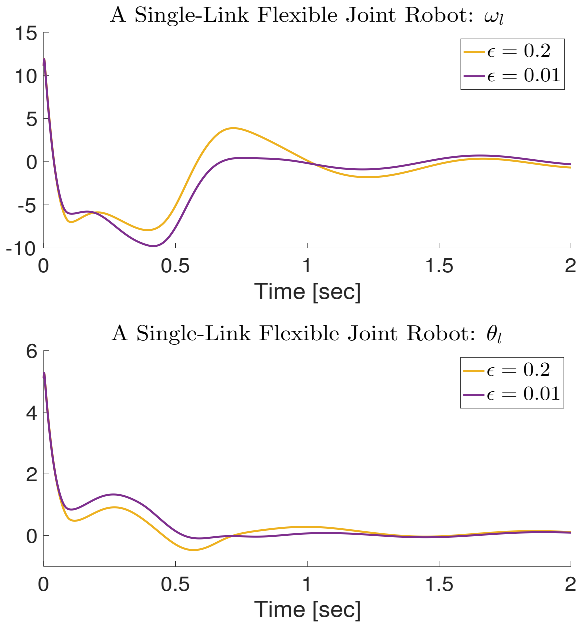

Figure 5.3: Performance of the estimation error for the angular rotation and angular velocity of the flexible joint robot link in (5.39).

5.4 Reduced-order circle criterion observer

Reduced-order observers are useful in a number of applications in which it might be more convenient to estimate only the unmeasured states. Here, we deal with a reduced-order version of the circle criterion observer considered in Section 5.3. In this section, we follow the same approach as in the previous one to analyse the robustness, with respect singular perturbations, of the reduced-order circle criterion observer introduced in [Section 5.2, 9].

The class of plants we consider in this section is the same class of systems introduced in Section 5.3, but with an special output map. Then, we just make reference to the content in Section 5.3 and mention the differences considered for this new case. Here, we focus on analysing nonlinear singularly perturbed systems with a structure given by (5.20a)-(5.20b), and an output defined by

$$y = Ex, \tag{5.41}$$

where $E \in \mathbb{R}^{p\times n}$ is defined as $E := [\mathbb{I}\ \mathbf{0}]$ with $\mathbb{I}$ being a $p \times p$ identity matrix and $\mathbf{0}$ being a $p\times(n-p)$ zero matrix. The definition of the output implies that y consists of p elements of the state vector. We consider Assumptions 5.3 and 5.4 for this case too. It is observed that Assumptions 4.1 - 4.5 are only related to the system dynamics without considering the output so that they hold under Assumption 5.3 and 5.4.

5.4.1 Observer design

Since we already analysed the plant in Section 5.3, we now focus on the reduced-order circle criterion observer introduced in [Section 5.2, 9] when is used to estimate the slow states of the system. To design a reduced-order observer for the slow system (5.23), we need an extra assumption over the model (5.23).

Assumption 5.6. *There is a change of coordinates such that the slow state vector is given by $x = [y^T, x_o^T]^T$. Moreover, the slow system in the new coordinates is*

$$\dot{y} = A_1 x_o + G_1\gamma([\bar{F}_1, \bar{F}_2][y, x_o]^T) + \sigma_1(y, u), \tag{5.42a}$$

$$\dot{x}_o = A_2 x_o + G_2\gamma([\bar{F}_1, \bar{F}_2][y, x_o]^T) + \sigma_2(y, u), \tag{5.42b}$$

where the linear terms in y are incorporated in the nonlinearities $\sigma_1(y, u)$, and $\sigma_2(y, u)$, and $\bar{F}_1$ and $\bar{F}_2$ are matrices of appropriate dimensions.

By following the design process given in [9], we have that the estimate of the unmeasured variable x_o is obtained via $\chi = x_o + Ny$, where $N \in \mathbb{R}^{(n-p)\times p}$ is to be designed. The following auxiliary subsystem is constructed from the definition of χ,

$$\dot{\chi} = (A_2 + NA_1)\chi + (G_2 + NG_1)\gamma(\bar{F}_2\chi + (\bar{F}_1 - \bar{F}_2 N)y) + \tilde{\sigma}(y, u), \tag{5.43}$$

where $\tilde{\sigma}(y, u) = N\sigma_1(y, u) + \sigma_2(y, u) - (A_2 + NA_1)y$. The reduced-order observer is designed by considering the auxiliary system (5.43). Then, the observer dynamics are given by

$$\dot{\hat{\chi}} = (A_2 + NA_1)\hat{\chi} + (G_2 + NG_1)\gamma(\overline{F}_2\hat{\chi} + (\overline{F}_1 - \overline{F}_2N)y) + \tilde{\sigma}(y, u). \tag{5.44}$$

The estimate of x_o, i.e., the output of the observer is given by $\hat{x}_o = \hat{\chi} - Ny$, which agrees with our framework relaying on Remarks 4.2, 4.4 and 4.7. The estimation error is defined as $e = x_o - \hat{x}_o = \chi - \hat{\chi}$ which verify that this design method fits our theory since conditions in Remark 4.2 hold. It follows that the error dynamics are given by

$$\begin{aligned}\dot{e} = (A_2 + NA_1)e + (G_2 + NG_1)[&\gamma(\overline{F}_2\chi + (\overline{F}_1 - \overline{F}_2N)y) \\ &- \gamma(\overline{F}_2(\chi - e) + (\overline{F}_1 - \overline{F}_2N)y)].\end{aligned} \tag{5.45}$$

It can be proven that y can be removed from the error dynamics (5.45) such that it becomes a function that only depends on e, see [Section 5.2, 9]. The alternative representation of (5.45) is

$$\dot{e} = (A_2 + NA_1)e + (G_2 + NG_1)\psi(t, \overline{F}_2e), \tag{5.46}$$

where $\psi(\cdot,\cdot) = \gamma(\overline{F}_2\chi + (\overline{F}_1 - \overline{F}_2N)y) - \gamma(\overline{F}_2(\chi - e) + (\overline{F}_1 - \overline{F}_2N)y)$. To verify Assumption 4.7, we consider the Lyapunov function $V_3(e) = e^T\hat{P}e$, where $\hat{P} = \hat{P}^T > 0$ is different from the the matrix considered in Section 5.3. Here, the matrix $\hat{P}$ is obtained by solving the following LMI

$$\begin{bmatrix}(A_2 + NA_1)^T\hat{P} + \hat{P}(A_2 + NA_1) + \hat{\nu}I & \hat{P}(G_2 + NG_1) + \overline{F}_2^T\Lambda \\ (G_2 + NG_1)^T\hat{P} + \Lambda\overline{F}_2 & 0\end{bmatrix} \leq 0, \tag{5.47}$$

where $\Lambda > 0$ is a diagonal matrix and an observer design parameter, and $\hat{\nu} > 0$ is also an observer design parameter. It follows from [9] that

$$\frac{\partial V_3}{\partial e} f_e(e) \leq -\hat{\nu}|e|^2, \tag{5.48}$$

when the LMI (5.47) is satisfied. Note that $f_e(e) = (A_2 + NA_1)e + (G_2 + NG_1)\psi(t, \overline{F}_2e)$ in (5.48). Therefore, we conclude that Assumption 4.7 is satisfied with $\underline{\alpha}_{V_3}(|e|) = \lambda_{\min}\{\hat{P}\}|e|^2$ and $\overline{\alpha}_{V_3}(|e|) = \lambda_{\max}\{\hat{P}\}|e|^2$, $\zeta_2 = \hat{\nu}$, $\alpha_{V_3}(|e|) = |e|$, and $\hat{\zeta}_2 = 2|\hat{P}|$. We know from Remark 4.5 that it is not mandatory to check Assumption 4.8 when the reduced-order observer relies on Remark 4.2. For this case, Assumption 4.8 can be verified by using the Lyapunov function $V_3(\hat{\chi})$ as a Lyapunov function for the observer dynamics (5.44). Moreover, Assumption 4.9 holds globally with $\alpha_y(|(x,\xi)|) = 2|E|\}|(x,\xi)|$ since the output does not depend on the singular perturbation parameter. Similar to the full-order observers case, we do not need Assumptions 4.6, 4.8 and 4.9 and Corollary 4.2 since the error dynamics completely capture the performance of the states of the observer.

Under the proposed approach, the observer designed for the reduced model must be implemented on the full system. Then, we have that the error dynamics when the observer is applied to the true system are given by

$$\begin{aligned}\dot{e} = (A_2 + NA_1)e + B_o\xi + (G_2 + NG_1)[\gamma(\overline{F}_2\chi + (\overline{F}_1 - \overline{F}_2N)y + F_2\xi) \\ - \gamma(\overline{F}_2(\chi - e) + (\overline{F}_1 - \overline{F}_2N)y + F_2\xi)],\end{aligned} \tag{5.49}$$

where B_o is a matrix of appropriate dimensions that agrees with the unmeasured state. Then, it follows that Assumption 4.10 is satisfied with $b_6 = 2|\hat{P}||B_0|$, while the rest of the constants and functions of u are zero. Since Assumption 4.1 - 4.10 hold, we conclude that our framework applies to the class of systems and the reduced-order observer considered in this section. We summarise the content of this section in the next corollary which is an immediate consequence of Theorem 4.1 and Corollary 4.3.

Corollary 5.3. *Consider the singularly perturbed plant (5.42), the circle criterion observer (5.44) and the error dynamics (5.49). If Assumptions 4.3, 4.5 and 5.3 - 5.6 hold, there exists a positive definite matrix* $\hat{P}$ *and* $\hat{\nu} > 0$*, such that for any* $\Delta > 0$, $\Delta_{u_1} > 0$, $\Delta_{u_2} > 0$ *and* $\mu > 0$*, there exists* $T^* > 0$ *and* $\varepsilon^* > 0$ *such that*

$$|e(t)| \le \sqrt{\frac{\lambda_{max}(\hat{P})}{\lambda_{min}(\hat{P})}}|e_0| \exp\left(-\frac{\hat{\nu}}{2\lambda_{max}(\hat{P})}t\right) + \mu, \tag{5.50}$$

for all $\varepsilon \in (0,\varepsilon^*)$, $|(x_0,\xi_0,\chi_0,e_0)| \le \Delta$, $\|u\|_\infty \le \Delta_{u_1}$, $\|\dot{u}\|_\infty \le \Delta_{u_2}$, *and* $t \ge \varepsilon T^* + t_0$. *Furthermore, for the given* $\Delta > 0$, $\Delta_{u_1} > 0$, $\Delta_{u_2} > 0$ *and* $\mu > 0$*, there exists* $T^* > 0$ *and* $\varepsilon^*_{\mathcal{L}_2} > 0$ *such that*

$$\int_{t_0}^{t} |e(\tau)|^2 d\tau \le \frac{2\lambda_{max}(\hat{P})}{\hat{\nu}}|e_0| + \mu(t - t_0). \tag{5.51}$$

for all $\varepsilon \in (0,\varepsilon^*_{\mathcal{L}_2})$, $|(x_0,\xi_0,\chi_0,e_0)| \le \Delta$, *for any input satisfying* $\|u\|_\infty \le \Delta_{u_1}$, $\|\dot{u}\|_\infty \le \Delta_{u_2}$, $|u|_{\mathcal{L}_2} \le \Delta_{u_1}$, $|\dot{u}|_{\mathcal{L}_2} \le \Delta_{u_2}$ *and for all* $t \ge \varepsilon T^* + t_0$.

5.4.2 Simulation results

In this section, we consider an academic example taken from [9] to evaluate the performance of the reduced circle criterion observer when used to estimate the slow states of singularly perturbed system. Let us consider the nonlinear system with linear fast dynamics described by the following model

$$\dot{x}_1 = x_2 + x_1^2, \tag{5.52a}$$

$$\dot{x}_2 = x_2 + x_3 \exp(x_2) + u, \tag{5.52b}$$

$$\dot{x}_3 = 2u, \tag{5.52c}$$

$$\varepsilon \dot{z} = x_1 - z, \tag{5.52d}$$

$$y = z. \tag{5.52e}$$

By setting $\varepsilon = 0$, we obtain that the reduced order system is given by

$$\dot{x}_1 = x_2 + x_1^2, \tag{5.53a}$$

$$\dot{x}_2 = x_2 + x_3 \exp(x_2) + u, \tag{5.53b}$$

$$\dot{x}_3 = 2u, \tag{5.53c}$$

$$y = x_1. \tag{5.53d}$$

We assume that $x_o = [x_2, x_3]^T$ so that the reduced system can be written in the form of (5.42) with the following matrices

$$A_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad G_1 = 0, \quad G_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad F_1 = 0, \quad F_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and with $\sigma_1(y, u) = y^2$ and $\sigma_2(y, u) = [u, 2u]^T$. By solving the LMI (5.47), we design a reduced-order observer with the following structure

$$\dot{\hat{\chi}}_2 = -\hat{\chi}_2 + \hat{\chi}_3 - \exp(\hat{\chi}_2 + 2y) + (-2y^2 - y + u), \tag{5.54a}$$

$$\dot{\hat{\chi}}_3 = -\hat{\chi}_2 + (-y^2 - 2y + 2u), \tag{5.54b}$$

$$\hat{x}_2 = \hat{\chi}_2 + 2y, \tag{5.54c}$$

$$\hat{x}_3 = \hat{\chi}_3 + y. \tag{5.54d}$$

We design a PD controller to guarantee that $x(t)$ remains bounded for all time. Then, we implement the observer (5.54) on the original plant (5.52). The simulation results for this numerical example are shown in Figure 5.4.

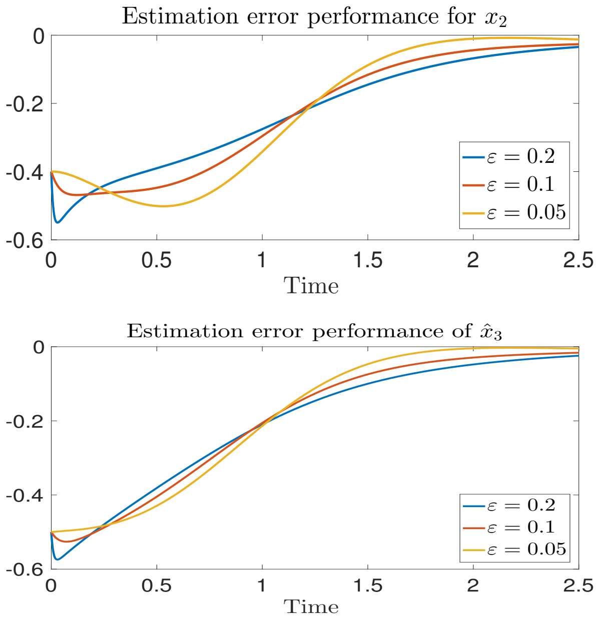

Figure 5.4: Estimation error performance for the states x_2 and x_3.

5.5 High-gain observer with limited gain power

We now analyse the class of singularly perturbed systems which has a structure such that the reduced (slow) model takes the form in which results from [14] can be applied to design a higher-order observer. Note that this class of plants is covered by the general model (4.1). Moreover, available results in [68] do not cover the class of systems and the observer considered in this section. Consider the class of systems with the following form

$$\dot{\mathbf{x}} = f_s(\mathbf{x}, \mathbf{z}), \tag{5.55a}$$

$$\varepsilon\dot{\mathbf{z}} = f_f(\mathbf{x}, \mathbf{z}), \tag{5.55b}$$

$$y = h(\mathbf{x}, \mathbf{z}) + w(t), \tag{5.55c}$$

where the state vector $\mathbf{x} \in \mathbf{X} \subset \mathbb{R}^n$ corresponds to the slow state, $\mathbf{z} \in \mathbf{Z} \subset \mathbb{R}^m$ is the fast state, $y \in \mathbb{R}^p$ is the measured output variable, $w \in \mathbb{R}$ is the measurement noise which belongs to $\mathcal{L}_\infty \cap \mathcal{L}_2$, and $\varepsilon > 0$ is the singular perturbation parameter of the process. We now check our assumptions for the class of systems represented by (5.55). Since the system does not consider inputs, Assumption 4.1 trivially holds. To obtain the lower dimensional systems, we set $\varepsilon = 0$ such that the system (5.55) is restricted to the slow manifold $0 = f_f(\mathbf{x}, \mathbf{z})$.

Assumption 5.7. *The algebraic equation* $0 = f_f(\mathbf{x}, \mathbf{z})$ *has a solution* $H(\mathbf{x})$ *which can be obtained analytically.*

Then, Assumption 4.2 holds by virtue of Assumption 5.7. Note that the reduced system is given by

$$\dot{\mathbf{x}} = f_s(\mathbf{x}, H(\mathbf{x})), \tag{5.56a}$$

$$y_s = h(\mathbf{x}, H(\mathbf{x})) + w(t). \tag{5.56b}$$

Assumption 5.8. *The reduced system (5.56) is input-to-state practical stable, such that there exists a Lyapunov ISpS function that satisfies Assumption 4.3. Moreover, there exists a transformation* $x = \phi_x(\mathbf{x})$*, such that the reduced system (5.56) can be written as*

$$\dot{x} = A_n x + B_n \psi(x), \tag{5.57a}$$

$$y = C_n x + w(t), \tag{5.57b}$$

where $\psi(\cdot)$ *is a locally Lipschitz function, and* (A_n, B_n, C_n) *is a triplet in "prime form" of dimension* n*, that is*

$$A_n = \begin{pmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ 0 & 0_{1\times(n-1)} \end{pmatrix}, \quad B_n = \begin{pmatrix} 0_{(n-1)\times 1} \\ 1 \end{pmatrix}, \quad C_n = \begin{pmatrix} 1 & 0_{1\times(n-1)} \end{pmatrix}.$$

The system (5.57) is defined on the set $\mathbb{X} \subset \mathbb{R}^n$ *where* $\mathbb{X} = \phi_x(\mathbf{X})$.

Note that the ISpS condition in Assumption 5.8 can be checked either in (5.56) or (5.57). It follows that Assumption 4.3 holds by virtue of Assumption 5.8. We have that the boundary layer system is given by

$$\frac{d\xi}{d\tau} = f_f(\mathbf{x}, \xi + H(\mathbf{x})) \tag{5.58}$$

where $\tau = t/\varepsilon$ is the fast-time scale and $\xi = \mathbf{z} - H(\mathbf{x})$.

Assumption 5.9. *There is a Lyapunov function* $W(\xi)$ *such that it can be proven that the boundary layer system is asymptotically stable. Furthermore, the full system (5.55) satisfies the interconnection conditions in Assumption 4.5.*

We require Assumption 5.9 to hold due to the generality of the boundary layer system (5.58). It follows from Assumption 5.9 that Assumptions 4.4 and 4.5 are satisfied.

5.5.1 Observer design

We now consider the high-gain observer with limited gain power proposed in [14] with the following dynamics

$$\begin{aligned} \dot{\chi}_i &= A\chi_i + N\chi_{i+1} + D_2(\ell)K_i\hat{e}_i, \quad i = 1, \dots, n-2, \\ &\vdots \\ \dot{\chi}_{n-1} &= A\chi_{n-1} + B\psi_s(\hat{x}) + D_2(\ell)K_{(n-1)}\hat{e}_{n-1}, \end{aligned} \tag{5.59}$$

where (A, B, C) is a triplet in prime form of dimension 2, $\chi = \mathrm{col}(\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{2n-2}$ is the state of the observer with $\chi_i \in \mathbb{R}^2$, $K_i = (k_{i1}, k_{i2})^T$ are the gains to be designed, $D_2(\ell) = \mathrm{diag}(\ell, \ell^2)$ with ℓ being the high gain parameter, $\hat{x} = L_1 x$ is the output of the observer with $L_1 = \mathrm{blkdiag}(\underbrace{C, \dots, C}_{(n-2)\ \text{times}}, I_2)$, $\hat{e}_1 = y - C\chi_1$, $\hat{e}_i = B^T\chi_{i-1} - C\chi_i$ $(i = 2, \dots, n-1)$, and $\psi_s(\cdot)$ is an appropriate saturated version of $\psi(\cdot)$. Note that state of the observer has a dimension of $2n-2$ so that the redundancy of the observer is used to obtain tow estimates with the asymptotic properties of the standard high-gain observer. Since the output of the observer is a linear map, it follows that Assumption 4.4 trivially holds. We now define the estimation error as $e := \hat{x} - x$. It can be show that the error dynamics are given by

$$\dot{e} = \ell L_1 M L_1^- e + L_1[\ell^{-(n-1)}(B_{2n-2}\Delta\psi_\ell(e, x) + w_\ell(t))], \tag{5.60}$$

where L_1^- is the left inverse of L_1, $\Delta\psi_\ell(\cdot,\cdot) = \psi_s(e+x) - \psi(x)$, $w_\ell(\cdot) = \ell^n \overline{K}_1 w(\cdot)$ with $\overline{K}_1 = \mathrm{col}(K_1, 0, \dots, 0)$, and

$$M = \begin{pmatrix} E_1 & N & 0 & & \cdots & \cdots & 0 \\ Q_2 & E_2 & N & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & Q_i & E_i & N & \ddots & \vdots \\ \vdots & & & \ddots & Q_{n-2} & E_{n-2} & N \\ 0 & \cdots & \cdots & \cdots & 0 & Q_{n-1} & E_{n-1} \end{pmatrix},$$

with $E_i = \begin{pmatrix} -k_{i1} & 1 \\ -k_{i2} & 0 \end{pmatrix}$, $Q_i = \begin{pmatrix} 0 & k_{i1} \\ 0 & k_{i2} \end{pmatrix}$, $N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Since the solutions evolve in a compact set and $\psi(\cdot)$ is locally Lipschitz, it follows that $\psi(\cdot)$ is uniformly Lipschitz in $\mathbf{X}$ and $\psi_s(\cdot)$ is bounded. Moreover, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|\ell^{-(n-1)}\Delta\psi_\ell(e,x)| \leq \delta_1|e|, \quad \text{and} \quad |\ell^{-(n-1)} w_\ell(t)| \leq \delta_2|\ell w(t)|,$$

for all $e \in \mathbb{R}^n$, $x \in \mathbf{X}$ and $\ell \geq 1$. Now, let $P_e = P_e^T$ be such that $P_e M_L + M_L^T P_e = -\mathbb{I}$ where $M_L = L_1 M L_1^-$ and $\mathbb{I}$ is the identity matrix. Consider $V_3(e) = e^T P e$ as a candidate Lyapunov function for (5.60). It can be proven that the time derivative of the aforementioned Lyapunov function along the solutions of (5.60) is bounded as follows

$$\dot{V}_3\big|_{(5.60)} \leq -\mathbf{a}_1\ell|e|^2 + \mathbf{a}_2|w|, \tag{5.61}$$

for all $\ell \geq 4\delta_1|P|$ where $\mathbf{a}_1$ and $\mathbf{a}_2$ are positive constants. Therefore, Assumption 4.7 holds with $\underline{\alpha}_{V_3}(|e|) = \lambda_{\min}\{P_e\}$, $\overline{\alpha}_{V_3}(|e|) = \lambda_{\max}\{P_e\}$, $\zeta_2 = \mathbf{a}_1\ell$, $\alpha_{V_3}(|e|) = |e|^2$, $\gamma_{V_3}(|w|) = \mathbf{a}_2|w|$, and $\hat{\zeta}_2 = 2|P_e|$. The Lyapunov analysis carried out in [14] considers the auxiliary variables $\eta_i = \ell^{2-i} D_2(\ell)^{-1}(\chi_i - \mathrm{col}(x_i, x_{i+1}))$ to construct a vector $\eta = \mathrm{col}(\eta_1, \dots, \eta_{n_1})$ which represent the estimation error of the two available estimates of x.

By using the Lyapunov function used in [14], it can be proven that (4.27) in Assumption 4.8 holds with $\alpha_{o_1}(|\chi_0|) = c_1\exp(-c_2\ell t)|\chi_0|$, $\alpha_{o_2}(||y||_\infty) = c_3||y||_\infty$ and $\alpha_{o_3}(||u||_\infty) = 0$ where $c_1 = \sqrt{\lambda_{\max}\{P\}/\lambda_{\min}\{P\}}$, $c_2 = d_1/2\lambda_{\max}\{P\}$, and $c_3 = \sqrt{\lambda_{\max}\{P\}/\lambda_{\min}\{P\}}d_2$ with $d_1 > 0$, $d_2 > 0$ and $P = P^T$ satisfying $PM + M^T P = -I$. Since we use a definition of $\mathcal{L}_2$ equivalent to ISS, the condition (4.28) can be concluded from the same analysis.

Note that the output of the system does not depend on the perturbation parameter.

Then, Assumption 4.9 must hold globally. Due to the generality of the output map, we assume that Assumption 4.9 holds. The generality of the system (5.55) does not allow to obtain a unique solution for the constants and functions in Assumption 4.10. The best we can do is to guarantee that Assumption 4.10 holds with $b_6 > 0$ and $b_7 \geq 0$ while $a_4 = 0$, $a_5 = 0$, $a_6 = 0$, $a_7 = 0$, $\gamma_5(\cdot) = 0$ and $\gamma_6(\cdot) = 0$.

Remark 5.3. *Sharper conclusions on the bounds in Assumption 4.10 can be concluded if we restrict the class of systems in (5.55). For instance, one can consider a linear fast dynamics.*

The following result is a consequence of a direct application of Theorem 4.1 to the class of systems and the higher-order observer considered in this section.

Corollary 5.4. *Consider the singularly perturbed plant (5.55), the higher-order observer (5.59) and the error dynamics (5.60). If Assumptions 5.7 - 5.9 hold, there exists a positive definite matrix* P_e*, and constants* $\mathbf{a}_1 > 0$ *and* $\mathbf{a}_2 > 0$*, such that for any* $\Delta > 0$*,* $\Delta_w > 0$ *and* $\mu > 0$*, there exists* $T^* > 0$ *and* $\varepsilon^* > 0$ *such that*

$$|e(t)| \leq \sqrt{\frac{\lambda_{\max}(P_e)}{\lambda_{\min}(P_e)}}|e_0| \exp\left(-\frac{\mathbf{a}_1 \ell}{2\lambda_{\max}(P_e)} t\right) + \sqrt{\frac{8\lambda_{\max}(P_e)}{\mathbf{a}_1 \ell \lambda_{\min}(P_e)} \mathbf{a}_2 \left\|w\left[t_0, t\right]\right\|^2} + \mu, \qquad (5.62)$$

for all $\varepsilon \in (0, \varepsilon^*)$, $|(x_0, \xi_0, \chi_0, e_0)| \leq \Delta$, $\|w\|_\infty \leq \Delta_w$*, and* $t \geq \varepsilon T^* + t_0$*. Furthermore, for the given* $\Delta > 0$*,* $\Delta_w > 0$ *and* $\mu > 0$*, there exists* $T^* > 0$ *and* $\varepsilon^*_{\mathcal{L}_2} > 0$ *such that*

$$\int_{t_0}^{t} |e(\tau)|^2 d\tau \leq \frac{2\lambda_{\max}(P_e)}{\mathbf{a}_1 \ell}|e_0| + \frac{2\mathbf{a}_2}{\mathbf{a}_1 \ell}\int_{t_0}^{t} |w(\tau)|^2 d\tau + \mu(t - t_0). \qquad (5.63)$$

for all $\varepsilon \in (0, \varepsilon^*_{\mathcal{L}_2})$, $|(x_0, \xi_0, \chi, e_0)| \leq \Delta$*, for any* $\|w\|_\infty \leq \Delta_w$, $|w|_{\mathcal{L}_2} \leq \Delta_w$ *and for all* $t \geq \varepsilon T^* + t_0$.

5.5.2 Simulation results

Let us consider the Van der Pol oscillator with a fast sensor dynamics described by the following mathematical model

$$\begin{aligned} \ddot{\eta} &= -\alpha^2 \eta + \beta(1 - \eta^2)\dot{\eta}, \\ \varepsilon \dot{z} &= \eta - z, \\ y &= z + w(t), \end{aligned} \qquad (5.64)$$

where $y \in \mathbb{R}$ is the available sensor measurement (output), α, β are uncertain constant parameters, and $w(t)$ is a bounded measurement noise. By setting $\varepsilon = 0$, the reduced system is obtained

$$\begin{aligned}\ddot{\eta}_s &= -\alpha^2\eta_s + \beta(1-\eta_s^2)\dot{\eta}_s,\\ y_s &= \eta_s + w(t).\end{aligned} \tag{5.65}$$

At $w(t) = 0$, we construct a high-gain nonlinear observer with limited gain power for the reduced order system in (5.65). To do so, the system must be rewritten in the canonical observability form as follows

$$\begin{aligned}\dot{x} &= Ax + B\psi(x),\\ y &= Cx + \nu(t),\end{aligned} \tag{5.66}$$

where $x = [x_1, x_2, x_3, x_4, x_5]^T$ is the vector of time derivatives of η_s, $\psi(x)$ is a locally Lipschitz function, and the matrices (A, B, C) in the prime form are defined as follows

$$A := \begin{bmatrix} 0_{4\times1} & \mathbb{I}_4 \\ 0 & 0_{1\times4}\end{bmatrix},\ B := \begin{bmatrix} 0_{4\times1} \\ 1\end{bmatrix},\ C := \begin{bmatrix} 1 & 0_{1\times4}\end{bmatrix}.$$

We design the high-gain limited power observer in (5.59) for the reduced system (5.66) and perform simulations. The simulation parameters and initial conditions were taken from [14]. The implemented observer produces redundant state estimates, but we have only consider one of them since the results are quite similar for both asymptotic estimates. In Figure 5.5, the simulation results are shown. On the top, the estimation error for the state x_1 is shown, several values of the perturbation parameter ε were tested. During simulations, it was observed that for large ε the estimation error for x_1 oscillates between -0.4 and 0.4, and it never converges to zero. It is an expected behaviour because of the nature of the Van der Pol system. On the bottom, the estimation error for the state x_2 is presented, in this case when ε grows up the estimation error still converges, but the convergence rate becomes slow. It is evident when the sensor dynamics are fast enough (small ε), such fast dynamics have no significant impact on the convergence properties of the estimation error, especially in the first state.

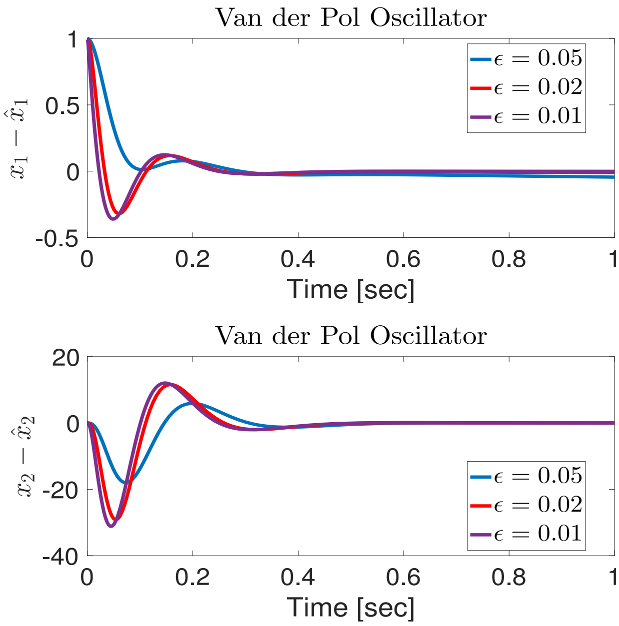

Figure 5.5: Estimation error performance for x_1 and x_2.

5.6 Conclusions of the chapter

We demonstrated that the observer design framework presented in Chapter 4 covers the class of nonlinear singularly perturbed systems and the nonlinear observer considered in [68]. Moreover, we show that our findings apply to other three classes of nonlinear singularly perturbed systems and three nonlinear observers of general dimension that can be designed for the reduced order models of these systems. These nonlinear observers are not covered by existing results in the literature. Furthermore, we illustrated the applicability of the theoretical design framework by presenting numerical examples and simulation results for each observer. Although we have checked that our assumptions in Chapter 4 hold for many nonlinear systems and observers, we have not included all of them here.

Part III

Parameter and State Estimation of Nonlinear Systems with Slowly Time-varying Parameters

IN THIS part of the thesis, we present a new estimation technique for parameter and state estimation of nonlinear systems with slowly time-varying parameters. The new algorithm is a generalisation of existing works on parameter and state estimation under the supervisory framework [25]. The proposed technique is able to deal with estimation of constant and slowly time-varying parameters. Moreover, the multi-observer approach is natural to singularly perturbed systems and can be used to estimate the full state of the plant. In Figure III.1, we show a schematic representation of the multi-observer approach.

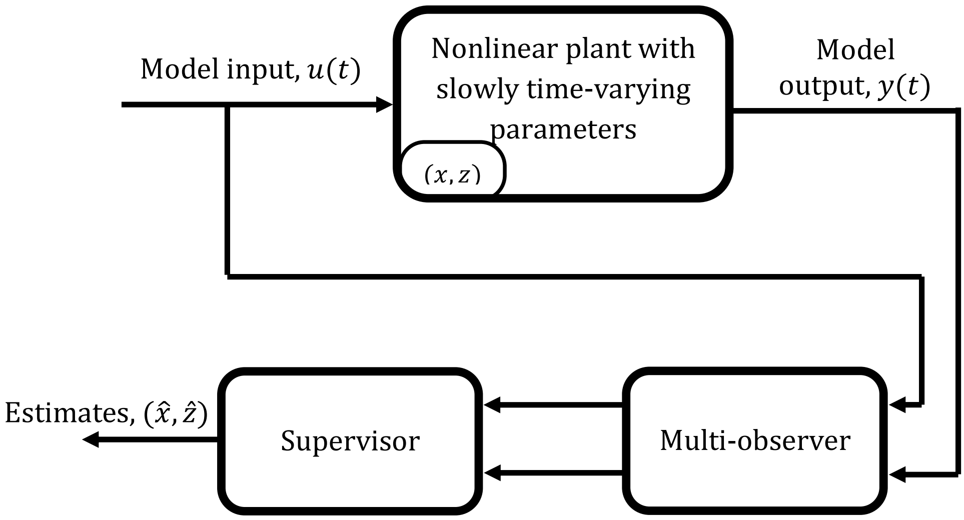

Figure III.1: Multi-observer approach for parameter and state estimation.

This part of the thesis contains Chapter 6 and Chapter 7. In Chapter 6, we first make a review on the existing theory related to the multi-observer approach under the supervisory framework for systems with unknown constant parameters. Then, we introduce a modified static sampling policy and a new and novel dynamic sampling policy which generalises the multi-observer approach to the case when the unknown parameter is slowly time-varying. We state and prove that the new sampling policies can achieve arbitrarily small parameter and state estimation errors.

In Chapter 7, we present simulation results to illustrate the performance of our proposed technique. Moreover, we demonstrate that the new dynamic sampling policy can deal with more general problems than the existing multi-observer technique in [25]. We show via simulations that the multi-observer for nonlinear systems with slowly time-varying parameters can be used for parameter and state estimation of systems with constant parameters in the presence of noisy measurements. We present simulation results for both vanishing and non-vanishing noise cases. Furthermore, we illustrate via simulations that our technique can deal with nonlinear systems with discontinuous slowly time-varying parameters with appropriate dwell-time between discontinuities. Finally, we demonstrate on a practical example that the our proposed approach can be implemented on singularly perturbed systems to estimate the full-state of the plant.

Chapter 6

Parameter and State Estimation of Systems with Slowly Time-Varying Parameters

This chapter delivers a multi-observer technique based on the supervisory framework for nonlinear autonomous plants with unknown slowly time-varying parameters. Here, we generalise existing results on the multi-observer approach for estimation of systems with unknown constant parameters. First, we study the convergence properties of the multi-observer technique when a static sampling policy is used to construct the bank of observers. Then, we present a new dynamic sampling policy which is capable of addressing more general problems than results in [25]. We present convergence results for the parameter and state estimation errors in which we show that both estimates can be made arbitrarily small when the norm of the time derivative of the unknown parameter is sufficiently small.

6.1 Introduction

THE SUPERVISORY control framework [15, 51, 90–92, 122] is a research area that deals with the stabilisation and output tracking of uncertain linear systems. The structure of the supervisory control can be summarised in two main components: 1) a multi-controller and 2) a supervisor that defines the switching among the controllers. The multi-controller is a family of candidate controllers parametrized by guesses of the uncertain parameters. On the other hand, the supervisor is constituted by a multi-estimator, a set of monitoring signals and a switching logic.

The multi-estimator is a bank of estimators that uses the input and output of the system to produce estimates of the unknown parameters and estimates of the output for a set of sample points of the unknown parameter. Then, by using the monitoring signals, a switching logic is generated to decide which controller would be active at each time instant. The multi-estimator under the supervisory framework for uncertain linear systems has motivated the multi-observer approach under the supervisory framework for parameter and state estimation of nonlinear autonomous systems with unknown constant parameters [25].

The multi-observer technique presented in [25] consists of a hybrid scheme for the parameter and state estimation of nonlinear systems where it is assumed that the unknown parameter vector is constant and belongs to a known compact set. The state observers are synthesized for a finite set of nominal sample parameter values to generate multiple state estimates. Then, a selection criterion chooses the estimate that gives the smallest difference between the measured and the estimated output by using a set of monitoring signals. This criterion provides state and parameter estimates at any given time instant. Note that the results in [25] can be regarded as the estimation of the fast and slow state by using the boundary layer system of a singularly perturbed plant. Hence, the multi-observer is natural to the singular perturbations framework.

Two sampling policies are presented in [25] to generate the parameter samples. The first one is a static sampling policy where a large number of samples is generated at the start of the algorithm and remain the same for all time. Since the number of observers determines the size of the neighbourhood around zero where the parameter and state estimation errors converge, the estimation errors can be made as small as desired by increasing the number of sample points and observers when a uniform sampling policy is used. Therefore, the static sampling policy would require a significant computational power. To counteract this problem, the second policy in [25] corresponds to a dynamic sampling policy where the parameter samples are periodically updated by using a zoom-in procedure to provide the same accuracy as the static policy while using a smaller number of observers.

When dealing with the problem of parameter and state estimation of nonlinear systems with slowly time-varying parameters, a possible solution is a multi-observer approach with a uniform sampled set with a large number of sample points. Here, we revisit the multi-observer approach with a static sampling policy from [25] and prove that, under appropriate modifications, it generates ultimately bounded parameter and state estimation errors. However, this sampling policy demands a significant computational power. Hence, a further generalisation of results in [25] requires a new dynamic sampling policy to guarantee accurate parameter and state estimates while having reduced computational requirements. The zoom-in procedure in the dynamic sampling policy introduced in [25] periodically reduces the size of the set where the samples are taken from and uses intersections with previous sets to obtain denser sampled sets at each iteration. The centre of the zoomed set is always moved to the last best parameter estimate so that the new set of samples are taken from a potential neighbourhood of the real parameter. However, this is not enough to deal with nonlinear systems with slowly time-varying parameters as these parameters may eventually leave the zoomed-in set after a finite time due to the reduction of the size of the set and the use of the intersection with previous sets.

A dynamic sampling policy able to deal with slowly time-varying parameters needs to incorporate a zoom-out procedure besides the zoom-in so that the sampled set can follow the moving parameter. We propose a periodical update of the sampled set inspired by the works in [25] and [80]. The zoom-in procedure follows the same structure as the dynamic sampling policy in [25]. On the other hand, the zoom-out procedure keeps the centre of the "box" at the same position of the last zoom-in and increases the size of the sampled set. In the zoom-out procedure, the new set is not intersected with any previous sampled set, the only needed condition is that the zoomed-out set is a subset of the known set where the unknown parameter lives. The new dynamic sampling policy consisting of zoom-in and zoom-out procedures is one of the contributions of this thesis. We provide convergence results of the parameter and state estimation errors as well as appropriate tuning algorithms that guarantee such convergence properties.

The outline of this chapter is as follows. In Section 6.2, we present the general class of plants that we consider in this chapter. In Section 6.3, we give a brief background on the multi-observer approach for nonlinear systems with unknown constant parameters. In Section 6.4, we address the parameter and state estimation problem of nonlinear systems with unknown slowly time-varying parameters. Finally, we present the conclusions of the chapter in Section 6.5.

6.2 Nonlinear plants with slowly time-varying parameters

The study of this chapter is done by considering a general class of nonlinear autonomous systems with slowly time-varying parameters. Hence, consider systems represented by the following model

$$\dot{\xi} = f(\xi, x(t), u), \tag{6.1a}$$

$$y = h(\xi, x(t), u), \tag{6.1b}$$

where $\xi \in \mathbb{R}^m$ is the state of the system, $y \in \mathbb{R}^p$ is the measured output, $u \in \mathbb{R}^r$ is the known input, and $x \in \mathbf{X}$ is an unknown time-varying parameter vector where $\mathbf{X} \subset \mathbb{R}^n$ is assumed to be a known compact set. Moreover, the derivative of the varying parameter is bounded by an small parameter $\varepsilon > 0$ so that $|\dot{x}(t)|_\infty \leq \varepsilon L_x$ for a fixed $L_x > 0$.

Assumption 6.1. *The maps f and h in (6.1) are assumed to be continuously differentiable.*

Assumption 6.2. *Consider the nonlinear system (6.1). For any $\Delta > 0$ and $\Delta_{u_1} > 0$, there exists $k_{A2} > 0$ such that for all $|\xi(0)| \leq \Delta$, $\|u\|_\infty \leq \Delta_{u_1}$ and $t \geq 0$, the following holds*

$$|\xi(t)|_\infty \leq k_{A2}. \tag{6.2}$$

Assumption 6.2 implies that the solutions to (6.1) are bounded for any bounded initial conditions and bounded inputs. This assumption is common within the estimation context since the estimation of nonlinear systems with unbounded solutions is a complicated task to solve. Moreover, a number of physical systems have solutions that are uniformly bounded. Note that $k_{A2} > 0$ does not need to be known in order to implement the estimation algorithms presented below. Before presenting our results on parameter and state estimation for nonlinear systems in the form of (6.1), we summarise results from [25].

6.3 Multi-observer for nonlinear systems with unknown constant parameters

In this section, we overview results in [25]. Note that those results can be related to the singular perturbations framework in the sense that they can be regarded as the estimation of the slow and fast states by using the boundary layer model. Let us consider the system

$$\dot{x} = 0, \tag{6.3a}$$

$$\dot{\xi} = f(\xi, x, u), \tag{6.3b}$$

$$y = h(\xi, x), \tag{6.3c}$$

where $\xi \in \mathbb{R}^m$ is the state of the system, $y \in \mathbb{R}^p$ is the measured output, $u \in \mathbb{R}^r$ is the known input, and $x \in \mathbf{X}$ is an unknown constant parameter vector where $\mathbf{X} \subset \mathbb{R}^n$ is assumed to be a known compact set.

Remark 6.1. *The nonlinear system (6.3) represents the particular case when $\varepsilon = 0$ in the nonlinear model (6.1). Moreover, the system (6.3) can be regarded as the boundary layer model of a singularly perturbed system as it has the same structure as the system (1.9).*

We assume that the system (6.3) satisfies Assumptions 6.1 and 6.2. We now present the multi-observer approach for parameter and state estimation introduced in [25]. The methodology explained below consists of two main units: a bank of observers (multi-observer) which generates potential state estimates and the supervisor which chooses one observer and one potential parameter estimate at any given time. The multi-observer under the supervisory framework introduced in [25] is depicted by Figure 6.1. Next, we will describe each component of this estimation technique.

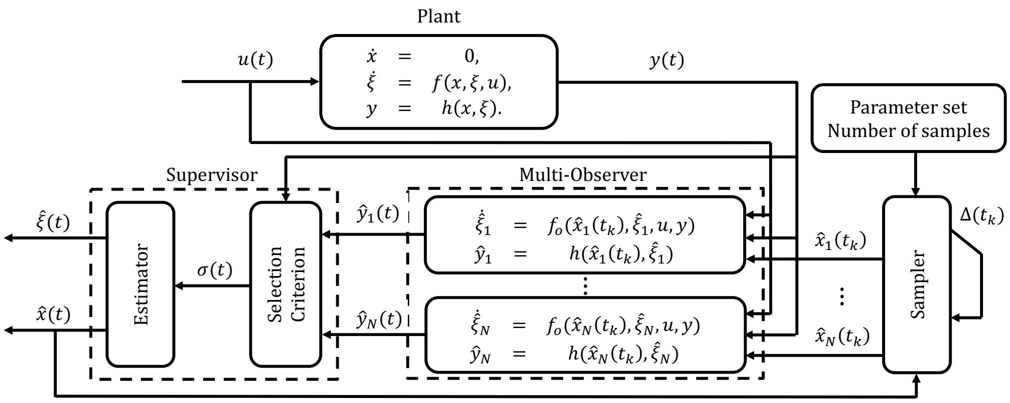

Figure 6.1: Multi-observer approach under the supervisory framework [25].

6.3.1 Static sampling policy

The multi-observer technique requires of finite number of sample points of the known set $\mathbf{X}$. So, consider the nonlinear system (6.3) where $x \in \mathbf{X}$ is a fixed unknown parameter. We select $N \in \mathbb{N}_{\geq 1}$ parameter values $\hat{x}_i$, for $i \in \{1, \dots, N\}$, from $\mathbf{X}$ to form the sampled set $\widehat{\mathbf{X}} = \{\hat{x}_1, \hat{x}_2, \dots, \hat{x}_N\}$. The selection of the samples is done such that

$$\max_{x \in \mathbf{X}} \left\{ d\left(x, \widehat{\mathbf{X}}\right) \right\} \to 0 \text{ as } N \to \infty. \tag{6.4}$$

Note that results in [25] do not consider the max notation, but our condition in (6.4) cover the one in [25]. Since the set $\mathbf{X}$ is assumed to be compact, it can be embedded in a hypercube. Hence, condition (6.4) can be guaranteed by employing a uniform sampling for instance. Although the static sampling policy is an easy way to address the parameter and state estimation problem under the multi-observer approach, it has an important disadvantage as we would need a large number of sample points and observers to generate accurate parameter and state estimates. This translates into a high computational cost as the observers must run in parallel. For systems with fixed unknown parameters, the static sampling policy may perform well without requiring a significant computational power. We next present the multi-observer.

Multi-observer

A state observer is designed for each $\hat{x}_i \in \widehat{\mathbf{X}}$, for $i \in \{1, \dots, N\}$. We assume that each of these observers is robust with respect to parameter errors in Assumption 6.4. For the system (6.3), consider the following multi-observer

$$\dot{\hat{\xi}}_i = f_o(\hat{\xi}_i, \hat{x}_i, u, y), \tag{6.5a}$$

$$\hat{y}_i = h(\hat{\xi}_i, \hat{x}_i, u), \tag{6.5b}$$

where $\hat{\xi}_i \in \mathbb{R}^m$, for $i \in \{1, \dots, N\}$, are the potential estimates of $\xi \in \mathbb{R}^m$, and $\hat{y}_i \in \mathbb{R}^p$ is the output estimate for each $\hat{x}_i \in \widehat{\mathbf{X}}$, for $i \in \{1, \dots, N\}$.

Assumption 6.3. *The map f_o in (6.5) is continuously differentiable. Furthermore, the solutions to (6.5) are unique and defined for all positive time, for all initial conditions, any input $u \in \mathbb{R}^r$, any system output $y \in \mathbb{R}^p$ and any sampled parameter $\hat{x}_i \in \widehat{\mathbf{X}}$, for $i \in \{1, \dots, N\}$.*

Denoting the state estimation error as $e_{\xi_i} := \hat{\xi}_i - \xi$, the output error as $e_{y_i} := \hat{y}_i - y$, and the parameter error as $e_{x_i} := \hat{x}_i - x$, we obtain the following state estimation error systems for the system (6.3) and the observer (6.5),

$$\dot{e}_{\xi_i} = \bar{f}_{e_i}(\xi, x, e_{x_i}, e_{\xi_i}, u), \tag{6.6a}$$

$$e_{y_i} = \bar{h}_e(\xi, x, e_{x_i}, e_{\xi_i}, u), \tag{6.6b}$$

for $i \in \{1, \dots, N\}$, where $\bar{f}_{e_i} = f_o(e_{\xi_i} + \xi, e_{x_i} + x, u, y) - f(\xi, x, u)$ and $\bar{h}_e = h(e_{\xi_i} + \xi, e_{x_i} + x, u) - h(\xi, x, u)$.

Observe that Assumption 6.3 alongside with Assumption 6.1 grant certain Lipschitz properties for the estimation error systems (6.6). Assumption 6.3 also guarantees there exists an estimate $\hat{\xi}_i \in \mathbb{R}^m$ for each $\hat{x}_i \in \widehat{\mathbf{X}}$, for $i \in \{1, \dots, N\}$, for all time $t \geq 0$. We now assume that the observers (6.5) are designed such that the following property holds.

Assumption 6.4. *There exist* $a_i > 0$, *for* $i \in \{1, \dots, 4\}$, $\lambda_0 > 0$, *a continuous non-negative function* $\tilde{\gamma} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}_{\geq 0}$ *with* $\tilde{\gamma}(0, \xi, u) = 0$ *for all* $\xi \in \mathbb{R}^m$, $u \in \mathbb{R}^r$ *such that for any* $\hat{x}_i \in \widehat{\mathbf{X}}$, *for* $i \in \{1, \dots, N\}$, *there exists a continuously differentiable function* $V_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$, *which satisfies the following for all* $e_{\xi_i} \in \mathbb{R}^m$, $\xi \in \mathbb{R}^m$, $x \in \mathbf{X}$, $u \in \mathbb{R}^r$

$$a_1 |e_{\xi_i}|_\infty^2 \leq V_i(x, e_{\xi_i}) \leq a_2 |e_{\xi_i}|_\infty^2, \tag{6.7}$$

$$\frac{\partial V_i}{\partial e_{\xi_i}} \bar{f}_{e_i}(\xi, x, e_{x_i}, e_{\xi_i}, u) \leq -\lambda_0 V_i(x, e_{\xi_i}) + \tilde{\gamma}(e_{x_i}, \xi, u). \tag{6.8}$$

Assumption 6.4 implies that the observer is robust with respect to small parameter errors on compact sets. When $e_{x_i} = 0$, (6.7) and (6.8) imply that the origin of the state estimation error system (6.6a) is globally exponentially stable. Observe that the negativity of the derivative would depend on how large is the magnitude of e_{ξ_i} with respect to e_{x_i} so that for large e_{x_i} the error dynamics (6.6a) become unstable.

Monitoring signals

We next define the monitoring signals $\mu_i(\cdot, \cdot)$, which are used to select the "best" estimates $\hat{\xi}_i$, $\hat{x}_i$, for $i \in \{1, \dots, N\}$, from the potential estimates that the multi-observer (6.5) produces. The signal associated with each observer is the exponentially weighted $\mathcal{L}_2$-norm [122] of the output error defined as

$$\mu_i(t_1, t_2) = \int_{t_1}^{t_2} \exp(-\lambda(t_2 - s)) |e_{y_i}(s)|_\infty^2 ds, \tag{6.9}$$

for $i \in \{1, \dots, N\}$, where $\lambda > 0$ is a design parameter. For the static sampling policy presented in this section, the monitoring signals (6.9) can be implemented as the following linear filters

$$\dot{\mu}_i(0, t) := -\lambda \mu_i(0, t) + |e_{y_i}(t)|_\infty^2, \tag{6.10}$$

for all $t \geq 0$ with $\mu_i(0, 0) = 0$, for $i \in \{1, \dots, N\}$. We now assume that the output error of each of the observers e_{y_i} satisfies the following property.

Assumption 6.5. *For any* $\Delta > 0$, $\Delta_{e_\xi} > 0$, *and* $\Delta_{u_1} > 0$, *there exist a class-*$\mathcal{K}_\infty$ *function* $\alpha_{e_y}(\cdot)$ *and a constant* $T_{A5} = T_{A5}(\Delta, \Delta_{e_\xi}, \Delta_{u_1}) > 0$ *such that for all* $\hat{x}_i \in \mathbf{X}$ *for* $i \in \{1, \dots N\}$,

$|\xi(0)| \leq \Delta$, $|e_{\xi_i}(0)| \leq \Delta_{e_\xi}$, *and* $\|u\|_\infty \leq \Delta_{u_1}$, *the corresponding solution to systems (6.3) and (6.6) satisfies*

$$\int_{t-T_{A5}}^{t} |e_{y_i}(\tau)|_\infty^2 d\tau \geq \alpha_{e_y}(|e_{x_i}|_\infty), \ \forall\, t \geq T_{A5}. \tag{6.11}$$

The inequality in Assumption 6.5 is known as the persistence of excitation (PE) condition that appears in identification and adaptive literature. This assumption holds when the output errors $e_{y_i} \in \mathbb{R}^p$, for $i \in \{1, \ldots, N\}$, satisfy the classical PE condition, see [Proposition 1, 25]. This assumption is used to guarantee the convergence properties of the multi-observer approach. Note that the excitation level grows as the norm of the parameter error increases. Then, the left-hand side of (6.11) gives quantitative information regarding the parameter estimation error.

The signal $\sigma : \mathbb{R}_{\geq 0} \to \{1, \ldots, N\}$ is used to choose a parameter estimate and an observer from (6.5) at every time instant. It is defined as

$$\sigma(t) := \arg \min_{i \in \{1,\ldots,N\}} \mu_i(t), \ \forall\, t \geq 0. \tag{6.12}$$

Note that no dwell time is guaranteed with this selection criterion, and rapid changes of the signal $\sigma(\cdot)$ are allowed. Nevertheless, these switches do not affect the dynamics of the observers (6.5) and system (6.3). Based on switching signal (6.12), the estimated parameter and the estimated state are given by

$$\hat{x}(t) := \hat{x}_{\sigma(t)}(t), \tag{6.13}$$

$$\hat{\xi}(t) := \hat{\xi}_{\sigma(t)}(t), \tag{6.14}$$

for all $t \geq 0$. The parameter and the state estimates are discontinuous in general because these signals switch among a finite family of continuous trajectories that are in general different at the switching instants.

Estimation error convergence result

Here, we first present Lemmas 6.1 and 6.2, which were stated and proved in [25] to define desirable properties of the error systems (6.6) and the monitoring signals (6.9). Lemma 6.1 states that the error systems (6.6) satisfy a local input-to-state stability property with respect to the parameter error e_{x_i}, for $i \in \{1, \ldots, N\}$. On the other hand, Lemma 6.2 guarantees that the monitoring signals (6.9) are lower and upper bounded by class-$\mathcal{K}_\infty$ functions of the parameter error e_{x_i}, for $i \in \{1, \ldots, N\}$.

Lemma 6.1. *Consider the system (6.3) and the error system (6.6). Let Assumptions 6.1 - 6.4 hold. Then, there exists* $\overline{k}_m > 0$ *and* $\overline{\lambda} > 0$ *such that for any* $\overline{\Delta} > 0$, $\overline{\Delta}_{e_\xi} > 0$, $\overline{\Delta}_u > 0$, *there exists* $\overline{\gamma}_{e_x}(\cdot) \in \mathcal{K}_\infty$ *such that the corresponding solutions to (6.6) satisfy, for* $i \in \{1, \ldots, N\}$

$$|e_{\xi_i}(t)|_\infty \leq \overline{k}_m \exp(-\overline{\lambda}t)|e_{\xi_i}(0)|_\infty + \overline{\gamma}_{e_x}(\|e_{x_i}\|_\infty), \tag{6.15}$$

for all $x, \hat{x}_i \in \mathbf{X}$, $|\xi(0)|_\infty \leq \overline{\Delta}$, $|e_{\xi_i}(0)|_\infty \leq \overline{\Delta}_{e_\xi}$, $\|u\|_\infty \leq \overline{\Delta}_u$, *and* $t \geq 0$.

Lemma 6.2. *Consider the system (6.3), the monitoring signals (6.9) and the error system (6.6). Let Assumptions 6.1 - 6.5 hold. For any* $\tilde{\Delta} > 0$, $\tilde{\Delta}_{e_\xi} > 0$, $\tilde{\Delta}_u > 0$, *and* $\nu > 0$, *there exist class-*$\mathcal{K}_\infty$ *functions* $\underline{\alpha}(\cdot)$ *and* $\overline{\alpha}(\cdot)$ *independent of* ν *and a constant* $T = T(\tilde{\Delta}, \tilde{\Delta}_{e_\xi}, \tilde{\Delta}_u, \nu) > 0$ *such that the monitoring signals* $\mu_i(0, t)$ *satisfy, for* $i \in \{1, \ldots, n\}$

$$\underline{\alpha}(|e_{x_i}|_\infty) \leq \mu_i(0, t) \leq \overline{\alpha}(|e_{x_i}|_\infty) + \nu, \tag{6.16}$$

for all $t \geq T$, $x, \hat{x}_i \in \mathbf{X}$, $|\xi(0)|_\infty \leq \tilde{\Delta}$, $|e_{\xi_i}(0)|_\infty \leq \tilde{\Delta}_{e_\xi}$, $\|u\|_\infty \leq \tilde{\Delta}_u$.

The result presented below in Theorem 6.1 was also stated and proved in [25] by using results from Lemmas 6.1 and 6.2. This theorem states that the estimates (6.13) and (6.14) are respectively guaranteed to converge to their true values x and ξ up to some given margins $\nu_{e_x} > 0$ and $\nu_{e_\xi} > 0$, provided that the distance between any point $x \in \mathbf{X}$ and the sampled set $\widehat{\mathbf{X}}$ is sufficiently small. Then, if the sampling is such that (6.4) holds, the accuracy of the estimates can be rendered as accurate as desired by increasing N (the number of samples and observers).

Theorem 6.1. *Consider system (6.3), the multi-observer (6.5), the monitoring signals (6.9), the selection criterion (6.12), and the estimates (6.13) and (6.14). Suppose Assumptions 6.1 - 6.5 are satisfied. For any* $\Delta > 0$, $\Delta_{e_\xi} > 0$, $\Delta_u > 0$ *and any margins* $\nu_{e_x} > 0$ *and* $\nu_{e_\xi} > 0$, *there exist* $\overline{K}_{e_\xi} > 0$, $\overline{K}_{e_x} > 0$, *and a sufficiently large* $N^* \in \mathbb{N}$ *such that for any* $N \geq N^*$, *the following holds for all* $|\xi(0)|_\infty \leq \Delta$, $|e_{\xi_i}(0)|_\infty \leq \Delta_{e_\xi}$ *for* $i \in \{1, \ldots, N\}$, $\|u\|_\infty \leq \Delta_u$, *and* $t \geq 0$

$$|e_{x_{\sigma(t)}}(t)|_\infty \leq \overline{K}_{e_x}, \tag{6.17}$$

$$|e_{\xi_{\sigma(t)}}(t)|_\infty \leq \overline{K}_{e_\xi}, \tag{6.18}$$

$$\limsup_{t\to\infty} |e_{x_{\sigma(t)}}(t)|_\infty \le \nu_{e_x}, \tag{6.19}$$

$$\limsup_{t\to\infty} |e_{\xi_{\sigma(t)}}(t)|_\infty \le \nu_{e_\xi}. \tag{6.20}$$

A significant disadvantage of the scheme presented in this section is the need for a sufficiently large N to guarantee that the estimates fall within the required margins. Therefore, the computational cost increases as one requires better estimates of x and ξ. Observe that one can use the multi-observer with a static sampling policy while working with systems with slowly time-varying parameters. We prove this later in Theorem 6.2.

6.3.2 Dynamic sampling policy (zoom-in approach)

In [25], it was shown that a reduced number of observers N can achieve the same accuracy as the approach described above if the sampled set is periodically updated by using a zoom-in procedure. We now present the general algorithm of this dynamic policy since it has inspired the new algorithm presented in this thesis. We introduce the hypercube $\overline{\mathbf{X}}$ which satisfies that $\mathbf{X} \subset \overline{\mathbf{X}}$ and discretise it with $N \in \mathbb{N}$ sample points where N is generated by [Theorem 2, 25]. The sampled set is denoted as

$$\widehat{\mathbf{X}} := \{\hat{x}_1, \ldots, \hat{x}_N\} \mid \hat{x}_i \in \overline{\mathbf{X}} \text{ for } i \in \{1, \ldots, N\}. \tag{6.21}$$

The sample points are generated such that the following property is verified

$$\max_{x\in\mathbf{X}} \left\{ d\left(x, \widehat{\mathbf{X}}\right) \right\} \le \pi(\Delta, N), \tag{6.22}$$

where $\pi(\cdot,\cdot) \in \mathcal{KL}$ satisfies that $\pi(\Delta, N) \le \Delta$. Observe that $\pi(s,r) = \min\left\{ s, \frac{s}{r^{\frac{1}{p}}} \right\}$ ensures (6.22), see [25].

When implementing a dynamic sampling policy, we work with time-varying sets $\overline{\mathbf{X}}(t_k)$ and $\widehat{\mathbf{X}}(t_k) = \{\hat{x}_1(t_k), \ldots, \hat{x}_N(t_k)\}$, where t_k, for $k \in \mathbb{N}$, is the updating time satisfying

$$t_{k+1} - t_k = T_d, \tag{6.23}$$

with $T_d > 0$ being a design parameter. The full algorithm and the generation of $\overline{\mathbf{X}}(t_k)$ are as explained in the following.

Algorithm 6.1. *Assume* $\overline{\mathbf{X}}$ *is an hypercube centred at some known* $x_c \in \mathbb{R}^n$ *and of edge length* $2\Delta > 0$ *such that* $\mathbf{X} \subset \overline{\mathbf{X}}$ *. Let the zooming-in parameter* $a \in (0,1)$ *be given. Let* N

be the number of samples and T_d *be the sampling time. These parameters are generated by [Theorem 2, 25]. In view of (6.23), let* $t_k := kT_d$*, for* $k \in \mathbb{N}$.

1. *Set* $k = 0$*. Let* $x_c(t_0) = x_c$ *and* $\Delta(t_0) = \Delta$.
2. *Generate the sampled set* $\widehat{\mathbf{X}}(t_k)$ *by using (6.21) and (6.22).*
3. *Design a state observer for (6.3) for each* $\hat{x}_i(t_k) \in \widehat{\mathbf{X}}(t_k)$*, for* $i \in \{1, \cdots, N\}$ *and* $k \in \mathbb{N}$

$$\dot{\hat{\xi}}_i = f_o(\hat{\xi}_i, \hat{x}_i(t_k), u, y),\ \forall\, t \in [t_k, t_{k+1}), \tag{6.24a}$$

$$\hat{y}_i = h(\hat{\xi}_i, \hat{x}_i(t_k), u), \tag{6.24b}$$

$$\hat{\xi}_i(t_k) = \hat{\xi}_i(t_k^-), \tag{6.24c}$$

where $\hat{\xi}_i \in \mathbb{R}^m$ *and* $\hat{y}_i \in \mathbb{R}^p$ *are the state and the output estimates. The monitoring signals (6.9) are implemented as follows, for* $i \in \{1, \ldots, N\}$ *and* $k \in \mathbb{N}$

$$\begin{aligned} \dot{\mu}_i(t_k, t) &= -\lambda\mu_i(t_k, t) + |e_{y_i}(t)|^2,\ \forall\, t \in [t_k, t_{k+1}), \\ \mu_i(t_k, t_k) &= 0. \end{aligned} \tag{6.25}$$

The selection criterion signal is as follows, for $k \in \mathbb{N}$

$$\sigma(t) := \mathit{arg} \min_{i \in \{1,\ldots,N\}} \mu_i(t_k, t),\ \forall\, t \in [t_k, t_{k+1}). \tag{6.26}$$

4. *Generate the new set to be sampled as follows*

$$x_c(t_{k+1}) = \hat{x}_{\sigma(t_{k+1}^-)}(t_{k+1}^-), \tag{6.27}$$

$$\Delta(t_{k+1}) = a\Delta(t_k), \tag{6.28}$$

$$\overline{\mathbf{X}}(t_{k+1}) = \mathbf{X}(x_c(t_{k+1}), \Delta(t_{k+1})) \cap \overline{\mathbf{X}}(t_k) \cap \cdots \cap \mathbf{X}(t_0). \tag{6.29}$$

5. *Let* $k = k + 1$*. Then, go to step 2.*

When using the above sampling policy in our setting, the slowly time-varying parameters may eventually leave the sampled set which reduces its size due to the zoom-in procedure. Hence, the parameter estimation error cannot be reduced arbitrarily even if we use a large number of observers. We next demonstrate and illustrate this on an example of a neural mass model with slowly time-varying parameters.

6.3.3 Case study: A neural mass model with slowly time-varying parameters

Here, we illustrate that Algorithm 6.1 may lead to arbitrarily large estimation errors when dealing with systems with unknown slowly time-varying parameters. The parameter estimation error can be as large as the diameter of the set in which the unknown parameter lives. To demonstrate that the error cannot be arbitrarily reduced when implementing the Algorithm 6.1, we consider the neural mass model used as an example in [25]. So, let us consider the class of nonlinear systems in the following form

$$\dot{z} = A(x(t))z + G(x(t))\gamma(Hz) + B(x(t))\sigma(u,y), \tag{6.30a}$$

$$y = C(x(t))z, \tag{6.30b}$$

where $x \in \mathbf{X} \subset \mathbb{R}^n$, $z \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^u$, $\gamma : \mathbb{R}^n \to \mathbb{R}^s$ and $\sigma : \mathbb{R}^r \times \mathbb{R}^p \to \mathbb{R}^q$. The matrices $A(x(t))$, $B(x(t))$, $C(x(t))$ and $G(x(t))$ are continuous in x on the known compact set $\mathbf{X}$. The neural mass model has the form of (6.30) with the following matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -a^2 & -2a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -a^2 & -2a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -b^2 & -2b \end{bmatrix}, \quad G(x(t)) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ x_1(t)ac_2 & 0 \\ 0 & 0 \\ 0 & x_2(t)bc_4 \end{bmatrix},$$

$$B(x(t)) = \begin{bmatrix} 0 & x_1(t)a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1(t)a & 0 & 0 \end{bmatrix}^T, \quad H = \begin{bmatrix} c_1 & 0 & 0 & 0 & 0 & 0 \\ c_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and $C = [0\ 0\ 1\ 0\ -1\ 0]$, where the parameters a, b, c_1, c_2, c_3 and c_4 are assumed to be known. The nonlinear terms in (6.30) are $\gamma = (S, S)$ and $\sigma(u, y) = (S(y), u)$ where the function S denotes the sigmoid function $S(v) := \frac{2e_0}{1+\exp[r(v_0,v)]}$ for $v \in \mathbb{R}$ with known constants e_0, v_0 and $r \in \mathbb{R}_{\geq 0}$. The states z_1, z_3 and z_5 are the membrane potential contributions of the pyramidal neurons, the excitatory and the inhibitory inter-neurons respectively, and z_2, z_4 and z_6 are their respective time derivatives. The unknown parameters x_1 and x_2 represent the synaptic gains of excitatory and inhibitory neuronal populations, respectively. For a further explanation of the neural mass model, its dynamics and its parameters, the reader is referred to [57].

While [25] assumes that the unknown vector parameter is constant, we assume the parameter is slowly time-varying and that it belongs to $\mathbf{X} := [4,8] \times [22,28]$. Moreover, we consider that

$$x_1 = \begin{cases} 6.5 + 0.01t & \text{if } x_1 \in (4,8), \\ 8 & \text{otherwise}, \end{cases} \tag{6.31a}$$

$$x_2 = \begin{cases} 25.5 + 0.015t & \text{if } x_2 \in (22,28), \\ 28 & \text{otherwise}, \end{cases} \tag{6.31b}$$

so that $|\dot{x}(t)| \leq \varepsilon$ where $\varepsilon = 0.015$. We consider the state observer introduced in [27] to construct the following multi-observer

$$\begin{aligned} \dot{\hat{z}}_i &= A(\hat{x}_i)\hat{z}_i + G(\hat{x}_i)\gamma(H\hat{z}_i + K(\hat{x}_i)(C\hat{x}_i - y)) + B(\hat{x}_i)\sigma(u, y) \\ &\quad + L(\hat{x}_i)(C(\hat{x}_i)\hat{z}_i - y), \end{aligned} \tag{6.32a}$$

$$\hat{y}_i = C(\hat{x}_i)\hat{z}_i, \tag{6.32b}$$

for $i \in \{1, \ldots, N\}$, where $K(\hat{x}_i)$ and $L(\hat{x}_i)$ are the observer gain matrices which are computed as described in the following. Suppose there exist real matrices $P_i = P_i^T > 0$, $M_i = \text{diag}(m_{i1}, \ldots, m_{in}) > 0$ and scalars ν_i, μ_i such that the following holds

$$\begin{bmatrix} \mathbf{A}(P_i, L(\hat{x}_i), \nu_i) & \mathbf{B}(P_i, M_i, K(\hat{x}_i)) & P_i \\ \star & \mathbf{E}(M_i) & 0 \\ \star & \star & -\nu_i \mathbb{I} \end{bmatrix} \leq 0, \tag{6.33}$$

where the elements are

$$\begin{aligned} \mathbf{A}(P_i, L(\hat{x}_i), \nu_i) &= P_i(A(\hat{x}_i) + L(\hat{x}_i)C(\hat{x}_i)) + (A(\hat{x}_i) + L(\hat{x}_i)C(\hat{x}_i))^T P_i + \nu_i \mathbb{I} \\ \mathbf{B}(P_i, M_i, K(\hat{x}_i)) &= P_i G(\hat{x}_i) + (H + K(\hat{x}_i)C(\hat{x}_i))^T M_i, \\ \mathbf{E}(M_i) &= -2M_i \text{diag}\left(\frac{1}{b_{\gamma_1}}, \cdots, \frac{1}{b_{\gamma_n}}\right), \end{aligned}$$

where $\mathbb{I}$ is the identity matrix and $b_{\gamma_k} \in \mathbb{R}^n \backslash 0$ is such that $\frac{\partial \gamma_k(\nu_k)}{\partial \nu_k} \leq b_{\gamma_k} < \infty$ for all $\nu_k \in \mathbb{R}$ where $\gamma = (\gamma_1, \ldots, \gamma_n)$.

To show that Algorithm 6.1 cannot achieve arbitrarily small parameter estimation errors when dealing with systems with unknown slowly-time varying parameters, we implement such algorithm to the neural mass model described above. In Figure 6.2, we show simulation results where we plotted with blue dots the sample points taken from the set **X**, the red dot is the estimated parameter vector and the yellow mark is the real unknown slowly time-varying parameter. To perform simulations, we assumed that the parameter moves from the center of **X** to the corner and stay there forever. In Table 6.1, we display the simulation parameters for the neural mass model.

$a = 100$	$b = 50$	$c_1 = 135$	$c_2 = 108$	$c_3 = 33.75$
$c_4 = 33.75$	$e_0 = 2.5$	$v_0 = 6$	$r = 0.56$	

Table 6.1: Simulation parameters for results displayed in Figure 6.2.

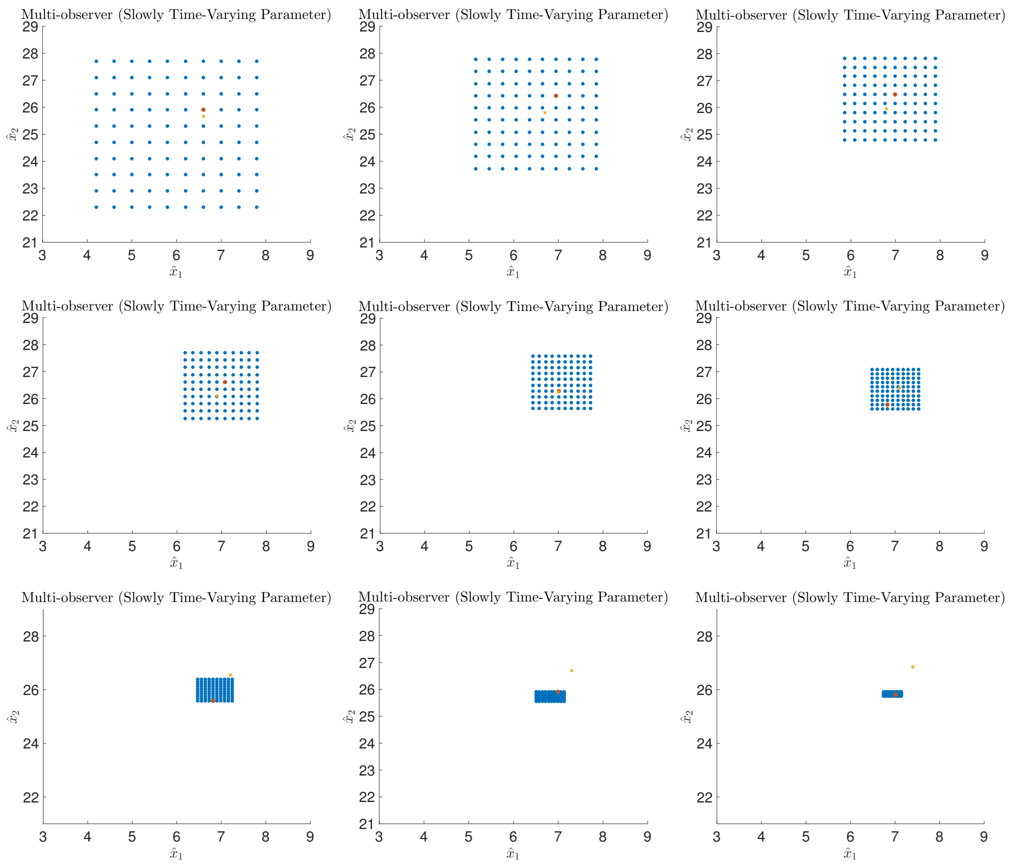

Figure 6.2: Yellow: Real parameter. Blue: Parameter sample points. Red: Parameter estimate.

Observe that in Figure 6.2 the parameter estimates obtained by using the dynamic sampling policy in Algorithm 6.1 are able to follow the real parameter just for a finite time. As the parameter vector moves away from its initial condition, Algorithm 6.1 cannot follow real parameter so that the parameter estimation error increases, and subse-

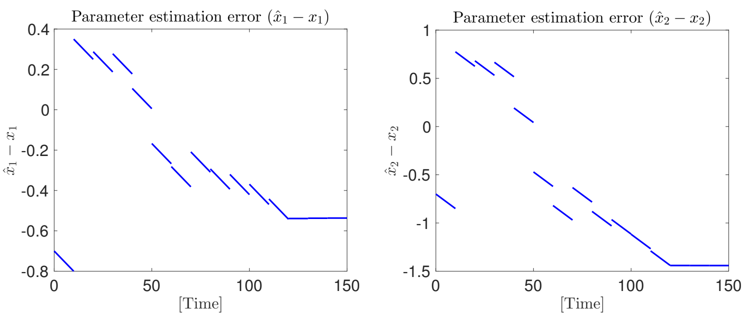

Figure 6.3: Parameter estimation errors for the neural mass model with slowly time-varying parameters.

quently, the state estimation error grows too. The performances of the parameter and state estimation errors are plotted in Figures 6.3 and 6.4. In Figure 6.4, we only included results for the states z_3 and z_5 as they exhibit the biggest state estimation errors.

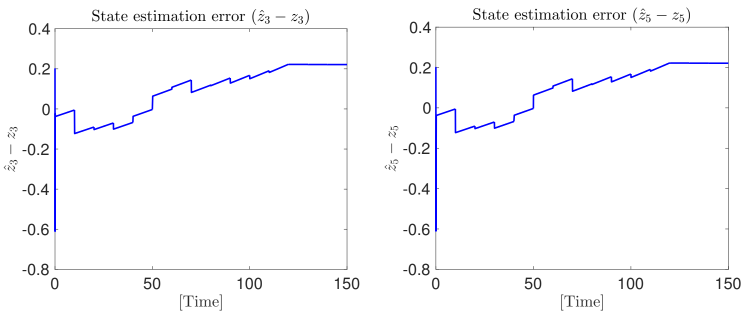

Figure 6.4: State estimation errors for z_3 and z_5 when using the dynamic sampling policy from [25].

The above results illustrate that we need a zoom-out procedure besides the zoom-in to allow the sampled set to recapture the varying parameter when it has left the set. This would permit to the parameter and state estimation errors to converge to an arbitrarily small neighbourhood around the origin. In the next section, we introduce a novel dynamic sampling policy that leads to a non-trivial generalisation of results in [25].

The dynamic policy from [25] may lose the parameter even when it is constant. However, the tuning conditions for the algorithm ensure that the parameter estimation error is sufficiently small even when the parameter is no longer in the sampled set. In the scenario of having noisy measurements, the dynamic sampling policy introduced in [25] cannot address small estimation errors if the sampled set loses the parameter. Although we do not analyse this scenario in here, we illustrate by simulation results that our proposed policy can address this problem in Chapter 7.

6.4 Multi-observer for nonlinear systems with unknown slowly varying parameters

In this section, we address the problem of parameter and state estimation of nonlinear systems with slowly time-varying parameters. We consider the class of plants defined by (6.1) where we have assumed that $x(t) \in \mathbf{X}$ is an unknown time-varying parameter vector where $\mathbf{X} \subset \mathbb{R}^n$ is a known compact set. Moreover, the derivative of the varying parameter is bounded as follows $|\dot{x}(t)|_\infty \leq \varepsilon L_x$ for a fixed $L_x > 0$ and a sufficiently small $\varepsilon > 0$.

As stated above, we propose here a novel dynamic sampling policy to tackle the main estimation problem of this chapter. However, before presenting the new dynamic sampling policy and proving a convergence result for it, we show that, under appropriate modifications, the static sampling policy in Section 6.3.1 can be used for parameter and state estimation of nonlinear systems with slowly time-varying parameters at the expense of a high computational cost. We then introduce a new dynamic sampling policy and state a convergence result for it. In Chapter 7, we show by simulations on a single neural mass model that the new sampling policy achieves the same accuracy as the static sampling policy with a reduced number of observers. Note that we present generalised versions of Lemmas 6.1 and 6.2 that can be applied to systems with slowly time-varying parameters.

6.4.1 Static sampling policy

The problem setting presented in this section is similar to the one in Section 6.3.1. Here, we consider those nonlinear systems with models defined by (6.1) where we have that $x(t) \in \mathbf{X}$ is an unknown slowly time-varying parameter. We select $N \in \mathbb{N}_{\geq 1}$ parameter values $\hat{x}_i$, for $i \in \{1, \dots, N\}$, from the known set $\mathbf{X}$ to form the sampled set $\widehat{\mathbf{X}} = \{\hat{x}_1, \hat{x}_2, \dots, \hat{x}_N\}$. The selection of the samples is done such that (6.4) holds.

Multi-observer

By following the multi-observer approach described in Section 6.3.1, we design a state observer for each $\hat{x}_i \in \widehat{\mathbf{X}}$, for $i \in \{1, \dots, N\}$. We consider that the multi-observer with the dynamics defined by (6.5) is synthesised for the system (6.1). Furthermore, we assume that each observer of the bank of observers in (6.5) satisfies Assumption 6.3.

As stated above (6.6), we define the state estimation error as $e_{\xi_i} := \hat{\xi}_i - \xi$, the output error as $e_{y_i} := \hat{y}_i - y$, and the parameter error as $e_{x_i} := \hat{x}_i - x$. It follows that state estimation error systems for the system (6.1) and the observer (6.5) are given by

$$\dot{e}_{\xi_i} = \bar{f}_{e_i}(\xi, x(t), e_{\xi_i}, e_{x_i}(t), u), \tag{6.34a}$$

$$e_{y_i} = \bar{h}_e(\xi, x(t), e_{\xi_i}, e_{x_i}(t), u), \tag{6.34b}$$

for $i \in \{1, \ldots, N\}$, where $\bar{f}_{e_i} = f_o(e_{\xi_i} + \xi, e_{x_i}(t) + x(t), u, y) - f(\xi, x(t), u)$ and $\bar{h}_e = h(e_{\xi_i} + \xi, e_{x_i}(t) + x(t), u) - h(\xi, x(t), u)$. Note that (6.34) has the same structure as (6.6) with the main difference that we now have that x is a slowly time-varying parameter so that e_{x_i} is also slowly time-varying.

The main aim of this section is to tackle the problem of parameter and state stimation of nonlinear systems with slowly time-varying parameters. Hence, since we want to prove a stronger result than those in [25], we need to assume that the state estimation errors satisfy stronger conditions than those in Assumption 6.4.

Assumption 6.6. *There exists $a_i > 0$, for $i \in \{1, \ldots, 4\}$, $\lambda_0 > 0$, a continuous non-negative function $\tilde{\gamma} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}_{\geq 0}$ with $\tilde{\gamma}(0, \xi, u) = 0$ for all $\xi \in \mathbb{R}^m$, $u \in \mathbb{R}^r$ such that for any $\hat{x}_i \in \widehat{\mathbf{X}}$, for $i \in \{1, \ldots, N\}$, there exists a continuously differentiable function $V_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$, which satisfies the following for all $e_{z_i} \in \mathbb{R}^m$, $\xi \in \mathbb{R}^m$, $x \in \mathbf{X}$, $u \in \mathbb{R}^r$*

$$a_1 |e_{\xi_i}|_\infty^2 \leq V_i(x, e_{\xi_i}) \leq a_2 |e_{\xi_i}|_\infty^2, \tag{6.35}$$

$$\frac{\partial V_i}{\partial e_{\xi_i}} \bar{f}_{e_i}(\xi, x, e_{\xi_i}, e_{x_i}, u) \leq -\lambda_0 V_i(x, e_{\xi_i}) + \tilde{\gamma}(e_{x_i}, \xi, u), \tag{6.36}$$

$$\left| \frac{\partial V_i}{\partial e_{\xi_i}} \right|_\infty \leq a_3 |e_{\xi_i}|_\infty, \quad \left| \frac{\partial V_i}{\partial x} \right|_\infty \leq a_4 |e_{\xi_i}|_\infty^2. \tag{6.37}$$

Note that the inequalities in (6.37) are needed to handle perturbations when the parameter is slowly time-varying. These conditions are standard when analysing the stability of nonlinear slowly varying systems [Section 9.6, 70].

The supervisor in the multi-observer approach consists of a set of monitoring signals and a selection criterion. For the case we are dealing with in this section, we consider the monitoring signals defined by (6.9). However, their implementation is changed with respect to the static sampling policy in Section 6.3.1. Here, we periodically reset the monitoring signals to zero to prevent them to increase unbounded. Furthermore, resetting the monitoring signals to zero allows to the selection criterion to choose an estimate based on the most recent data. We consider a finite time $T_d > 0$

so that $T_d = t_{k+1} - t_k$ where t_k is a resetting time of the monitoring signals. Hence, we implement the monitoring signals as follows, for $i \in \{1, \ldots, N\}$ and $k \in \mathbb{N}$

$$\begin{aligned} \dot{\mu}_i(t_k, t) &= -\lambda \mu_i(t_k, t) + |e_{y_i}(t)|^2, \ \forall\, t \in [t_k, t_{k+1}), \\ \mu_i(t_k, t_k) &= 0. \end{aligned} \tag{6.38}$$

We assume that the output error of each of the observers satisfies a persistence of excitation condition implied by a modified version of Assumption 6.5 where we let $e_{x_i} = e_{x_i}(t_k)$ and $\hat{x}_i = \hat{x}_i(t_k)$. So, let us consider the following assumption.

Assumption 6.7. *For any $\Delta > 0$, $\Delta_{e_\xi} > 0$, and $\Delta_{u_1} > 0$, there exist a class-$\mathcal{K}_\infty$ function $\alpha_{A7}(\cdot)$ and a constant $T_{A7} = T_{A7}(\Delta, \Delta_{e_\xi}, \Delta_{u_1}) > 0$ such that for all $\hat{x}_i(t_k) \in \mathbf{X}$, $i \in \{1, \ldots N\}$, $|\xi(0)| \le \Delta$, $|e_{\xi_i}(0)| \le \Delta_{e_\xi}$, and $\|u\|_\infty \le \Delta_{u_1}$, the corresponding solution to systems (6.3) and (6.6) satisfies*

$$\int_{t-T_{A7}}^{t} |e_{y_i}(\tau)|_\infty^2 d\tau \ge \alpha_{A7}(|e_{x_i}(t_k)|_\infty), \tag{6.39}$$

for all $t \ge t_k + T_{A7}$ for any $t_k \ge 0$.

The selection criterion in the supervisor is given by the piecewise constant function defined as follows

$$\sigma(t_{k+1}) := \arg \min_{i \in \{1,\ldots,N\}} \mu_i(t_k, t_{k+1}). \tag{6.40}$$

Estimation error convergence result

We now present the convergence results for the multi-observer technique when implemented on systems with slowly time-varying parameters by using a static sampling policy. We first present three useful results needed to prove the result in Theorem 6.2. In Lemma 6.3, we show that each of the error systems (6.34) satisfies a practical ISS property with respect to the parameter error e_{x_i}, for $i \in \{1, \ldots, N\}$. We then prove that Assumption 6.7 leads to a weaker persistence of excitation condition for the systems with slowly time-varying parameters. Furthermore, we prove that the monitoring signals $\mu_i(0, t)$, for $i \in \{1, \ldots, N\}$, defined by (6.9) are lower and upper bounded in Lemma 6.5 where the upper and lower bounds are different from those in [Lemma 2, 25]. These results are then used to conclude Theorem 6.2 which states a convergence result for the estimates obtained via the multi-observer (6.5) when used on nonlinear systems with slowly time-varying parameters.

Lemma 6.3. *Consider the system (6.1), and the estimation error systems (6.34). Let Assumptions 6.1 - 6.3, and 6.6 hold. Then, there exist* $k_{L1} > 0$, $\lambda_{L1} > 0$ *and* $\tilde{\varepsilon}^* > 0$ *such that for any* $\overline{\Delta} > 0$, $\overline{\Delta}_{e_\xi} > 0$ *and* $\overline{\Delta}_{u_1} > 0$, *there exists* $\gamma_L(\cdot) \in \mathcal{K}_\infty$ *such that the corresponding solutions to (6.34) satisfy, for* $i \in \{1, \dots, N\}$,

$$|e_{\xi_i}(t)|_\infty \leq k_{L1} \exp(-\lambda_{L1} t)\, |e_{\xi_i}(0)|_\infty + \gamma_L(\|e_{x_i}\|_\infty), \tag{6.41}$$

for all $\varepsilon \in (0, \tilde{\varepsilon}^*)$, $\hat{x}_i \in \mathbf{X}$, $|\xi(0)|_\infty \leq \overline{\Delta}$, $|e_{\xi_i}(0)|_\infty \leq \overline{\Delta}_{e_\xi}$, $\|u\|_\infty \leq \overline{\Delta}_{u_1}$, *and* $t \geq 0$.

Remark 6.2. *Observe that inequalities in (6.37) in Assumption 6.6 can be relaxed at the expense of concluding a practical input-to-state stability property in Lemma 6.3. Relaxing conditions in Assumption 6.6 to hold with general nonlinear* $\alpha(\cdot) \in \mathcal{K}$ *functions would allow using a larger class of observers. Note that this relaxation can be done as we have assumed that the system (6.1) has bounded solutions.*

The proof of Lemma 6.3 is presented in Appendix C.1. This result is used to prove Lemma 6.5 as well as the Theorem 6.2.

Lemma 6.4. *Consider the error systems (6.34) and let Assumptions 6.1 - 6.3, 6.6 and 6.7 hold. For any* $\underline{\Delta} > 0$, $\underline{\Delta}_{e_\xi} > 0$ *and* $\underline{\Delta}_{u_1} > 0$, *there exist a class-*$\mathcal{K}_\infty$ *function* $\alpha_L(\cdot)$, *a constant* $T_f = T_f(\underline{\Delta}, \underline{\Delta}_{e_\xi}, \underline{\Delta}_{u_1}) > 0$, $k_{PE} > 0$ *and* $\tilde{\varepsilon}^* > 0$, *such that the following holds, for* $i \in \{1, \dots N\}$,

$$\int_{t-T_f}^{t} |e_{y_i}(\tau)|_\infty^2 d\tau \geq \max\left\{\alpha_L(|e_{x_i}(t_k)|_\infty) - \varepsilon^2 k_{PE}, 0\right\}, \tag{6.42}$$

for all $t \geq t_k + T_f$, *for any* $t_k \geq 0$, $\varepsilon \in (0, \tilde{\varepsilon}^*)$, $|\xi(0)|_\infty \leq \Delta$, $|e_{\xi_i}(0)|_\infty \leq \Delta_{e_\xi}$, *and* $\|u\|_\infty \leq \Delta_{u_1}$.

The corresponding proof of Lemma 6.4 is presented in Appendix C.2. We have stated the result in Lemma 6.4 since it is needed to be able to prove the following lemma.

Lemma 6.5. *Consider the system (6.1), the monitoring signals (6.9) and the error system (6.34). Let Assumptions 6.1 - 6.3, 6.6 and 6.7 hold. For any* $\tilde{\Delta} > 0$, $\tilde{\Delta}_{e_\xi} > 0$, $\tilde{\Delta}_{u_1} > 0$ *and* $\nu > 0$, *there exist class-*$\mathcal{K}_\infty$ *functions* $\underline{\chi}(\cdot)$ *and* $\overline{\chi}(\cdot)$ *independent of* ν, $k_{LM} > 0$, *a constant* $T = T(\tilde{\Delta}, \tilde{\Delta}_{e_\xi}, \tilde{\Delta}_{u_1}, \nu) > 0$, $T_d \geq T$ *and* $\overline{\varepsilon}^* > 0$ *such that the monitoring signals* $\mu_i(t_k, t)$ *satisfy, for* $i \in \{1, \dots, n\}$

$$\max\left\{\underline{\chi}(|e_{x_i}(t_k)|_\infty) - \varepsilon^2 k_{LM}, 0\right\} \leq \mu_i(t_k, t) \leq \overline{\chi}(|e_{x_i}(t_k)|_\infty) + \nu, \tag{6.43}$$

for all $t \in [t_k + T, t_{k+1})$, $k \in \mathbb{N}$, *and for all* $\varepsilon \in (0, \bar{\varepsilon}^*)$, $x(t), \hat{x}_i \in \mathbf{X}$, $|\xi(0)|_\infty \leq \tilde{\Delta}$, $|e_{\xi_i}(0)|_\infty \leq \tilde{\Delta}_{e_\xi}$ *and* $\|u\|_\infty \leq \tilde{\Delta}_{u_1}$.

The proof of Lemma 6.5 is presented in Appendix C.3. We are now ready to state a convergence result for the parameter and state estimation errors for the case of having a nonlinear system with slowly time-varying parameters when using a static sampling policy. The proof of Theorem 6.2 uses results in Lemmas 6.3 - 6.5 to conclude the result. The proof of Theorem 6.2 is presented in Appendix C.4.

Theorem 6.2. *Consider system (6.1), the multi-observer (6.5), the monitoring signals (6.9), the selection criterion (6.12), and the static sampling policy. Let Assumptions 6.1 - 6.3, 6.6 and 6.7 hold. Then, for any* $\Delta > 0$, $\Delta_{e_\xi} > 0$, $\Delta_u > 0$, $\tilde{\nu}_{e_x} > 0$ *and* $\tilde{\nu}_{e_\xi} > 0$, *there exist* $\tilde{K}_{e_\xi} > 0$, $\tilde{K}_{e_x} > 0$, $\hat{\varepsilon}^* > 0$, *and a sufficiently large* $N^* \in \mathbb{N}$ *such that for any* $N \geq N^*$, *the following holds*

$$|e_{x_{\sigma(t)}}(t)|_\infty \leq \tilde{K}_{e_x}, \tag{6.44}$$

$$|e_{\xi_{\sigma(t)}}(t)|_\infty \leq \tilde{K}_{e_\xi}, \tag{6.45}$$

$$\limsup_{t\to\infty} |e_{x_{\sigma(t)}}(t)|_\infty \leq \tilde{\nu}_{e_x}, \tag{6.46}$$

$$\limsup_{t\to\infty} |e_{\xi_{\sigma(t)}}(t)|_\infty \leq \tilde{\nu}_{e_\xi}, \tag{6.47}$$

for all $\varepsilon \in (0, \hat{\varepsilon}^*)$, $|\xi(0)|_\infty \leq \Delta$, $|e_{\xi_i}(0)|_\infty \leq \Delta_{e_\xi}$ *for* $i \in \{1, \ldots, N\}$, $\|u\|_\infty \leq \Delta_u$, *and* $t \geq 0$.

Remark 6.3. *The static sampling policy can be used for parameter and state estimation of nonlinear systems with slowly time-varying parameters. However, it needs a large number of observers to guarantee arbitrarily small parameter and state estimation errors after a sufficiently large* $t \geq 0$. *Hence, the static sampling policy requires a significant computational power. We illustrate this through numerical simulation results in Chapter 7. The number of observers can be reduced by using the dynamic sampling policy presented in the next section while achieving the same accuracy as the static policy. This is also illustrated via simulations in Chapter 7.*

6.4.2 A novel dynamic sampling policy

We now introduce a new dynamic sampling policy designed to overcome the computational cost of the static method, see Theorem 6.2, and the inability of the dynamic sampling policy proposed in [25] to deal with slowly time-varying parameters. This new dynamic sampling policy is such that allows the implementation the multi-observer technique on systems with slowly time-varying parameters. This policy consists of a zoom-in and a zoom-out procedure inspired by the work in [80]. Results in this section generalise those in [25].

Algorithm 6.2. *Let* $\Delta_0 \geq 0$ *and* $x_c \in \mathbb{R}^n$ *be given such that* $\mathbf{X} \subset \mathbf{X}(x_c, \Delta_0)$. *Let* $a \in (0,1)$, $b > 1$, $c > 0$, $\delta_0 > 0$, *and* $\delta_1 \in (0, \delta_0)$, *and let* $N \in \mathbb{N}$ *and* $T_d > 0$, *where* a *and* b *are the zoom-in and zoom-out factors, respectively,* c *is a threshold for the zoom-out procedure,* δ *is a threshold for the monitoring signals,* N *is the number of samples and* T_d *is the sampling time. These parameters are generated by Theorem 6.3 presented below. In view of (6.23), let* $t_k := kT_d$, *for* $k \in \mathbb{N}$. *Moreover, let* $\hat{x}_0$ *be the initial condition for the parameter estimate and let* $m(t_k)$ *be a discrete variable which will take values in the set* {*'zoom-in', 'zoom-out'*} *with initial value* $m(t_0) =$ *'zoom-in'.*

1. *Set* $k = 0$. *Let* $x_c(t_0) = x_c$ *and* $\Delta(t_0) = \Delta_0$ *such that* $\mathbf{X} \subset \mathbf{X}(x_c(t_0), \Delta(t_0))$ *and define* $\overline{\mathbf{X}}(t_0) = \mathbf{X}(x_c(t_0), \Delta(t_0)) \cap \mathbf{X}$.
2. *Generate the sampled set* $\widehat{\mathbf{X}}(t_k)$ *by using (6.21) and (6.22).*
3. *Design a state observer for (6.3) for each* $\hat{x}_i(t_k) \in \widehat{\mathbf{X}}(t_k)$, *for* $i \in \{1, \cdots, N\}$ *and* $k \in \mathbb{N}$

$$\dot{\hat{\xi}}_i = f_o(\hat{\xi}_i, \hat{x}_i(t_k), u, y), \ \forall\, t \in [t_k, t_{k+1}), \tag{6.48a}$$

$$\hat{y}_i = h(\hat{\xi}_i, \hat{x}_i(t_k), u), \tag{6.48b}$$

$$\hat{\xi}_i(t_k) = \hat{\xi}_i(t_k^-), \tag{6.48c}$$

where $\hat{\xi}_i \in \mathbb{R}^m$ *and* $\hat{y}_i \in \mathbb{R}^p$ *are the state and the output estimates. The monitoring signals (6.9) are implemented as follows, for* $i \in \{1, \ldots, N\}$ *and* $k \in \mathbb{N}$

$$\begin{aligned} \dot{\mu}_i(t_k, t) &= -\lambda \mu_i(t_k, t) + |e_{y_i}(t)|^2, \ \forall\, t \in [t_k, t_{k+1}), \\ \mu_i(t_k, t_k) &= 0. \end{aligned} \tag{6.49}$$

The selection criterion signal is as follows, for $k \in \mathbb{N}$

$$\sigma(t_{k+1}) := \mathit{arg} \min_{i \in \{1,\ldots,N\}} \mu_i(t_k, t_{k+1}). \tag{6.50}$$

4. *Let* $\mu_{\sigma(t_{k+1})} = \min_{i\in\{1,\dots,N\}} \mu_i(t_k, t_{k+1})$, *and*

$$m(t_{k+1}) = \begin{cases} \textit{`zoom-in'} & \textit{if}\, \mu_{\sigma(t_{k+1})} < \delta_1, \\ \textit{`zoom-out'} & \textit{if}\, \mu_{\sigma(t_{k+1})} > \delta_0, \\ m(t_k) & \textit{if}\, \mu_{\sigma(t_{k+1})} \in [\delta_1, \delta_0]. \end{cases} \tag{6.51}$$

5. *Implement the following,*

- ***Zoom-in****: If* $m(t_{k+1})$ = *'zoom-in', let*

$$x_c(t_{k+1}) = \hat{x}_{\sigma(t_{k+1}^-)}(t_{k+1}^-), \tag{6.52}$$

$$\Delta(t_{k+1}) = a\Delta(t_k), \tag{6.53}$$

$$\overline{\mathbf{X}}(t_{k+1}) = \mathbf{X}(x_c(t_{k+1}), \Delta(t_{k+1})) \cap \overline{\mathbf{X}}(t_k). \tag{6.54}$$

- ***Zoom-out****: If* $m(t_{k+1})$ = *'zoom-out', let*

$$x_c(t_{k+1}) = x_c(t_k), \tag{6.55}$$

$$\Delta(t_{k+1}) = b \max\{\Delta(t_k), c\}, \tag{6.56}$$

$$\overline{\mathbf{X}}(t_{k+1}) = \mathbf{X}(x_c(t_{k+1}), \Delta(t_{k+1})) \cap \overline{\mathbf{X}}(t_0). \tag{6.57}$$

6. *Let* $k = k + 1$. *Then, go to step 2.*

We introduce some new notation. Note that for each $k \in \mathbb{N}_{\geq 1}$ we have $m(t_k) =$ 'zoom-in' or $m(t_k) =$ 'zoom-out' so that a zoom-in or a zoom-out is implemented at iteration k. Hence, there is a subsequence of intervals on which we zoom-in or zoom-out for a given initial condition. We introduce $k_j \in \mathbb{N}_{\geq 1}$ such that

$$\begin{aligned} m(t_k) &= \text{`zoom-in'}, &&\forall\, t_k \in [t_{k_{2j}}, t_{k_{2j+1}-1}], \\ m(t_k) &= \text{`zoom-out'}, &&\forall\, t_k \in [t_{k_{2j+1}}, t_{k_{2j+2}-1}], \end{aligned} \tag{6.58}$$

for $j \in \{0, \dots, \overline{N}\}$, with either finite $\overline{N} \in \mathbb{N}$ or $\overline{N} = \infty$ since there may be finitely or infinitely many switchings between zoom-in and zoom-out. For all our results, we will always let $k_0 = 1$, except when it is stated differently.

Estimation error convergence result

The multi-observer in Algorithm 6.2 is required to satisfy Assumption 6.3. Observe that the estimation error systems for (6.1) and (6.48) are as follows

$$\dot{e}_{\xi_i} = \overline{f}_{e_i}(\xi, x(t), e_{\xi_i}, e_{x_i}(t), u), \ \forall\, t \in [t_k, t_{k+1}), \tag{6.59a}$$

$$e_{y_i} = \overline{h}_e(\xi, x(t), e_{\xi_i}, e_{x_i}(t), u), \tag{6.59b}$$

for $i \in \{1, \cdots, N\}$ and $k \in \mathbb{N}$, where $\overline{f}_{e_i}$ and $\overline{h}_e$ are as defined after (6.6). The parameter estimation error is as defined above of (6.6), i.e. $e_{x_i} = \hat{x}_i - x$. Observe that the Algorithm 6.2 implies that $e_{x_i}(\cdot)$ are piecewise continuous functions with jumps (discontinuities) at each $t = t_k$, $k \in \mathbb{N}$. Although all variables on the right hand side of (6.59) depend on the time, we have highlighted only the time dependency of $x(t)$ and $e_{x_i}(t)$ to make a clear distinction respect to the error systems (6.6).

We now present a convergence result for the multi-observer technique when using the new dynamic sampling policy presented above for the case of having unknown time-varying parameters. Since our Algorithm 6.2 is such that $\hat{x}_i(t_k) \in \mathbf{X}$, for $i \in \{1, \dots, n\}$ and $k \in \mathbb{N}$, we are able to invoke Lemmas 6.3 and 6.5.

Theorem 6.3. *Consider the nonlinear system (6.1), the Algorithm 6.2, and the error systems (6.59). Let Assumptions 6.1 - 6.3, 6.6 and 6.7 hold. Then, for any given* $\hat{\Delta} > 0$, $\hat{\Delta}_{e_\xi} > 0$, $\hat{\Delta}_{u_1} > 0$, $\hat{\nu}_{e_x} > 0$, $\hat{\nu}_{e_\xi} > 0$, *zooming factors* $a \in (0, 1)$ *and* $b > 1$, *a constant* $c \in (0, \min\{\hat{\nu}_{e_x}, \gamma_L^{-1}(\hat{\nu}_{e_\xi})\}/2b\sqrt{n})$, $\delta_0 \in \left(0, \underline{\chi}((1-\theta)c)\right)$, *for* $\theta \in (0, 1)$, *and* $\delta_1 \in (0, \delta_0)$, *there exists* $\widehat{K}_{e_x} > 0$, $\widehat{K}_{e_\xi} > 0$, *sufficiently large* $T^* > 0$ *and* $N^* \in \mathbb{N}$ *such that for any* $T_d \geq T^*$ *and* $N \geq N^*$ *there exists* $\varepsilon^* > 0$, *constructed according to Algorithm 6.3 below, such that the following holds*

$$|e_{x\sigma(t)}(t)|_\infty \leq \widehat{K}_{e_x}, \tag{6.60}$$

$$|e_{\xi_{\sigma(t)}}(t)|_\infty \leq \widehat{K}_{e_\xi}, \tag{6.61}$$

$$\limsup_{t \to \infty} |e_{x\sigma(t)}(t)|_\infty \leq \hat{\nu}_{e_x}, \tag{6.62}$$

$$\limsup_{t \to \infty} |e_{\xi_{\sigma(t)}}(t)|_\infty \leq \hat{\nu}_{e_\xi}, \tag{6.63}$$

for all $\varepsilon \in (0, \varepsilon^*)$, $|\xi(0)|_\infty \leq \hat{\Delta}$, $|e_{\xi_i}(0)|_\infty \leq \hat{\Delta}_{e_\xi}$ *for* $i \in \{1, \dots, N\}$, $\|u\|_\infty \leq \hat{\Delta}_{u_1}$, *and* $t \geq 0$.

Algorithm 6.3. *(Construction of* $T^* > 0$, $N^* \in \mathbb{N}$, $N \geq N^*$, $T_d > 0$, *and* $\varepsilon^* > 0$*): Let* $\hat{\Delta} > 0$, $\hat{\Delta}_{e_\xi} > 0$, $\hat{\Delta}_{u_1} > 0$, $\hat{\nu}_{e_x} > 0$, $\hat{\nu}_{e_\xi} > 0$, $a \in (0, 1)$, $b > 1$, $c \in (0, \min\{\hat{\nu}_{e_x}, \gamma_L^{-1}(\hat{\nu}_{e_\xi})\}/2b\sqrt{n})$,

$\delta_0 \in \big(0, \underline{\chi}((1-\theta)c)\big)$, *for* $\theta \in (0,1)$, *and* $\delta_1 \in (0,\delta_0)$ *be given. Consider* $\gamma_L(\cdot) \in \mathcal{K}_\infty$ *and* $\tilde{\varepsilon}^* > 0$ *generated by Lemma 6.3,* $\underline{\chi}(\cdot), \overline{\chi}(\cdot) \in \mathcal{K}_\infty$*,* $k_{LM} > 0$ *and* $\overline{\varepsilon}^* > 0$ *generated by Lemma 6.5 and* $\pi(\cdot,\cdot) \in \mathcal{KL}$ *satisfying (6.22). Then, we have the following.*

1. *Define* $\hat{\eta} > 0$ *as follows*

$$\hat{\eta} := \min\left\{\hat{\nu}_{e_x}, \gamma_L^{-1}\left(\hat{\nu}_{e_\xi}\right)\right\}. \tag{6.64}$$

2. *Select* $\Delta_0 > 0$ *as stated in Algorithm 6.2,* $\Delta_2 \in (0,\Delta_0)$ *and* $\Delta_1 \in (\Delta_2,\Delta_0)$ *such that* $\Delta_2 = c$ *and* $\Delta_1 = 2bc\sqrt{n}$.
3. *Select* $T^* > 0$*,* $N^* \in \mathbb{N}$ *sufficiently large and* $T_d \geq T^*$ *such that Lemma 6.5 holds with* $\nu > 0$ *sufficiently small such that*

$$\underline{\chi}^{-1}(\overline{\chi}(\pi(s,N^*)) + 2\nu) \leq as, \tag{6.65}$$

   for all $s \in [\Delta_2,\Delta_0]$.
4. *Select* $N > 0$ *sufficiently large such that* $N \geq N^*$*,*

$$\overline{\chi}(\pi(\Delta_0,N)) + \nu \leq \delta_0, \tag{6.66}$$

   and

$$\overline{\chi}(\pi(\Delta_1,N)) + \nu < \delta_1. \tag{6.67}$$

   Recall that $0 < \delta_1 < \delta_0$.
5. *Define* $\varepsilon^* > 0$ *as follows*

$$\varepsilon^* := \min\{\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*, \varepsilon_4^*, \varepsilon_5^*, \varepsilon_6^*\}. \tag{6.68}$$

   where

$$\varepsilon_1^* = \tilde{\varepsilon}^*, \tag{6.69}$$

$$\varepsilon_2^* = \overline{\varepsilon}^*, \tag{6.70}$$

$$\varepsilon_3^* = \frac{\pi(\Delta_2,N)}{L_x T_d}, \tag{6.71}$$

$$\varepsilon_4^* = \sqrt{\frac{\nu}{k_{LM}}}, \tag{6.72}$$

$$\varepsilon_5^* = \sqrt{\frac{\underline{\chi}\left((1-\theta)\Delta_2\right) - \delta_0}{k_{LM}}}, \tag{6.73}$$

$$\varepsilon_6^* = \frac{(b-1)\Delta_2}{2L_x T_d}. \tag{6.74}$$

Remark 6.4. *A large number of sampling points* N*, leading to a large number of observers, would require significant computational resources. Hence, we aim to reduce the computational cost by having a time-varying set* $\overline{\mathbf{X}}(t_k)$ *and a reduced number of sample points and observers,* $N \in \mathbb{N}$*. To achieve better accuracy of the estimates, we implement a zoom-in procedure; however, when applying the multi-observer technique to systems with slowly time-varying parameters,* $x(t)$ *may occasionally exit the set* $\overline{\mathbf{X}}(t_k)$ *since* $x(t)$ *is slowly changing and the parameter sampled set is reducing its size. This was demonstrated via simulation results on a neural mass model in Section 6.3.3. Therefore, we need a zoom-out procedure to capture* $x(t)$ *within the set* $\overline{\mathbf{X}}(t_k)$ *if the parameter has left* $\overline{\mathbf{X}}(t_k)$*. Since it is not possible to directly know when* $x(t)$ *has left* $\overline{\mathbf{X}}(t_k)$*, we switch between zoom-in and zoom-out based on the monitoring signals which indirectly help to check if* $x(t)$ *is close to* $\widehat{\mathbf{X}}(t_k)$*.*

Remark 6.5. *It is always possible to ensure (6.65) since* $\underline{\chi}(\cdot), \overline{\chi}(\cdot) \in \mathcal{K}_\infty$ *and* $\pi(\cdot,\cdot) \in \mathcal{KL}$*. Furthermore, the choice of* δ_0 *and* Δ_2 *guarantees a positive numerator in the argument of the square root of the forth term of (6.68). Observe that we can always choose* $\delta_0 \in \left(0, \underline{\chi}\left((1-\theta)c\right)\right)$*, for* $\theta \in (0,1)$*,* $\delta_1 \in (0, \delta_0)$ *and* $N \geq \mathbb{N}^*$ *such that (6.66) and (6.67) hold due to the properties of the functions* $\overline{\chi}(\cdot)$ *and* $\pi(\cdot,\cdot)$ *and because* $\Delta_0 > \Delta_1$*.*

Remark 6.6. *The argument within the square root in (6.73) is always positive as* $\Delta_2 = c$ *and* $\delta_0 \in \left(0, \underline{\chi}\left((1-\theta)c\right)\right)$*, for* $\theta \in (0,1)$*.*

We now present two useful results that are the key ingredients for the proof of Theorem 6.3. We first study the behaviour of the Algorithm 6.2 when $m(t_k) =$ 'zoom-in' for all $t_k \in [t_{k_{2j}}, t_{k_{2j+1}-1}]$ in Lemma 6.6. Then, we provide a bound for the infinity norm of the parameter estimation error for the time intervals $[t_{k_{2j+1}}, t_{k_{2j+2}-1}]$, for $j \in \mathbb{N}$, in Lemma 6.7. Furthermore, we show in Lemma 6.7 that the construction of the observer parameters in Algorithm 6.3 guarantees that $k_{2j+2} - k_{2j+1} = 1$, for all $j \in \mathbb{N}$, which implies the zoom-out interval consists of one iteration and then the sampling policy switch to the zoom-in interval. These two results are concatenated in the proof of Theorem 6.3 to guarantee that, after the transient has terminated, the parameter estimation error becomes ultimately bounded. Then, we use this ultimate bound to show that the state estimates are ultimately bounded too. It is proven that the ultimate bounds can be made arbitrarily small if ε is sufficiently small and if the observer parameters are appropriately tuned.

Lemma 6.6. *Let conditions of Theorem 6.3 hold such that* $\hat{\eta} > 0$, $\Delta_0 > 0$, $\Delta_1 > 0$, $\Delta_2 > 0$, $T^* > 0$, $N^* \in \mathbb{N}$, $N \geq N^*$, $T_d > 0$, *and* $\varepsilon^* > 0$ *are generated by Algorithm 6.3. Consider* $t_k \in [t_{k_{2j}}, t_{k_{2j+1}-1}]$, *for* $j \in \mathbb{N}$. *Then, for all* $\varepsilon \in (0, \varepsilon^*)$, $|\xi(0)|_\infty \leq \hat{\Delta}$, $|e_{\xi_i}(0)|_\infty \leq \hat{\Delta}_{e_\xi}$ *for* $i \in \{1, \ldots, N\}$, $\|u\|_\infty \leq \hat{\Delta}_{u_1}$, *and* $t \geq 0$, *the following holds, for any* t_k

$$\Delta(t_{k-1}) \geq \Delta_2 \Longrightarrow x(t_k) \in \overline{\mathbf{X}}(t_k) \Longrightarrow m(t_k) = \text{'zoom-in'}, \tag{6.75}$$

and

$$x(t_k) \notin \overline{\mathbf{X}}(t_k) \Longrightarrow \Delta(t_{k-1}) < \Delta_2, \tag{6.76}$$

moreover

$$m(t_{k_{2j+1}}) = \text{'zoom-out'} \Longrightarrow x(t_{k_{2j+1}}) \notin \overline{\mathbf{X}}(t_{k_{2j+1}}) \Longrightarrow \Delta(t_{k_{2j+1}-1}) < \Delta_2. \tag{6.77}$$

Furthermore, for any $\Delta(t_{k_{2j}-1}) = \Delta_{in} \in (\Delta_2, \Delta_0]$, *the following holds*

$$\left| e_{x\sigma(t_k^-)}\left(t_k^-\right) \right|_\infty \leq \max\left\{ a^{k-k_{2j}+1}\Delta_{in}, \Delta_2 \right\}, \tag{6.78}$$

for all $t_k \in [t_{k_{2j}}, t_{k_{2j+1}-1}]$, *and moreover*

$$|e_{x\sigma(t^-_{k_{2j+1}-1})}(t^-_{k_{2j+1}-1})|_\infty < \Delta_2, \tag{6.79}$$

for all $j \in \mathbb{N}$.

The proof of Lemma 6.6 is presented in Appendix C.5. Since our construction guarantees that $m(t_{k_0}) =$ 'zoom-in', the result in Lemma 6.6 characterises the transient of the multi-observer approach when using the new dynamic sampling policy proposed in Algorithm 6.2. This result also delivers the appropriate bounds so that we are able to relate the end of a 'zoom-in' interval with the beginning of a 'zoom-out' interval.

Lemma 6.7. *Let conditions of Theorem 6.3 hold such that* $\hat{\eta} > 0$, $\Delta_0 > 0$, $\Delta_1 > 0$, $\Delta_2 > 0$, $T^* > 0$, $N^* \in \mathbb{N}$, $N \geq N^*$, $T_d > 0$, *and* $\varepsilon^* > 0$ *are generated by Algorithm 6.3. Assume there exists* $\Delta_{out} \in (0, \Delta_2)$ *such that* $\Delta(t_{k_{2j+1}-1}) \leq \Delta_{out}$ *for all* $j \in \mathbb{N}$. *Then, for all* $\varepsilon \in (0, \varepsilon^*)$,

$|\xi(0)|_\infty \le \hat{\Delta}$, $|e_{\xi_i}(0)|_\infty \le \hat{\Delta}_{e_\xi}$ *for* $i \in \{1,\ldots,N\}$, $\|u\|_\infty \le \hat{\Delta}_{u_1}$, *and* $t \ge 0$, *the following holds*

$$k_{2j+2} - k_{2j+1} = 1, \tag{6.80}$$

and the parameter estimation error satisfies

$$|e_{x\sigma(t^-_{k_{2j+2}-1})}(t^-_{k_{2j+2}-1})|_\infty < \Delta_1, \tag{6.81}$$

for all $j \in \mathbb{N}$.

The proof corresponding to Lemma 6.7 is presented in Appendix C.6. Observe that $\Delta_{out} \le \Delta_2$, $\Delta_1 \in (\Delta_2, \Delta_0)$, (6.79) and (6.81) imply that we can concatenate results from Lemmas 6.6 and 6.7. Therefore, these two lemmas are the key ingredients to obtain an ultimate bound of the parameter estimation error in proof of Theorem 6.3 presented in Appendix C.7.

Remark 6.7. *The proposed dynamic sampling policy can be used on systems with constant parameters and with noisy measurements. In Chapter 7, we demonstrate via simulations that our proposed technique can deal with the case when the noise vanishes after a finite time as well as with non-vanishing noise. For the case of non-vanishing noise, we illustrate that the state estimation errors exhibit an ISS behaviour with gain from the measurement noise. We present simulation results for these problems in Chapter 7 and a rigorous analysis is left for future work.*

6.5 Conclusions of the Chapter

In this chapter, we addressed the problem of parameter and state estimation of nonlinear systems with unknown slowly time-varying parameters. We tackled the problem by using a multi-observer approach under the supervisory framework. We proposed a novel dynamic sampling policy that uses zoom-in and zoom-out procedures to allow to the parameter estimates to follow a slowly time-varying parameter. This new dynamic sampling policy lead to a generalisation of existing results for parameter and state estimation in [25]. We rigorously proved that our proposed technique guarantees the parameter and state estimation errors can be made as small as desired if the slowly time-varying parameter moves sufficiently slow and if the observer is carefully tuned.

Chapter 7

Applications of multi-observer approach

In this chapter, we implement the multi-observer approach presented in Chapter 6 on a nonlinear autonomous system with slowly time-varying parameters. We demonstrate via simulations that our proposed technique can be used on systems with constant parameters and noisy measurements. We illustrate that our dynamic sampling policy works well when the plant has unknown discontinuous slowly time-varying parameters with a sufficiently large dwell-time. Furthermore, we present simulation results on full-state estimation of a singularly perturbed system.

7.1 Introduction

THE NEW DYNAMIC sampling policy presented in Chapter 6 generalises results in [25] to the case of having nonlinear systems with slowly time-varying parameters. Here, we present simulation results where we estimate the parameter and state of a neural mass model with unknown slowly time-varying parameters. We first revisit the case study presented in Chapter 6 and demonstrate that the static sampling policy can achieve arbitrarily small parameter and state estimation errors after a sufficiently large $t \geq 0$ if we use a large number of observers. We then illustrate through simulations that our proposed sampling policy in Algorithm 6.2 is able to deal with systems with unknown slowly time-varying parameters while using a reduced number of observers.

We also show via simulations that the proposed multi-observer approach under the supervisory framework can address more general problems when the new sampling policy is used. We present simulation results for the case when the neural mass model has unknown constant parameters with noisy measurements. We illustrate that our technique can address the problem of having vanishing and non-vanishing measurement noise. These are problems that cannot be addressed by results in [25]. We also show simulations for the case when the slowly time-varying parameter is discontinuous with a sufficiently large τ_d dwell-time, i.e. there is a τ_d time interval between consecutive discontinuities. Finally, we demonstrate via simulations that the Algorithm 6.2 can be used for the full-state estimation of singularly perturbed systems when the fast dynamics treat the slow states as slowly time-varying parameters. These scenarios are only presented via simulations and their theoretical study is left for future work.

7.2 Neural mass model

In this chapter, we present simulation results for the neural mass model studied in Section 6.3.3. Hence, consider the class of systems defined by

$$\dot{z} = A(x(t))z + G(x(t))\gamma(Hz) + B(x(t))\sigma(u, y), \tag{7.1a}$$

$$y = C(x(t))z, \tag{7.1b}$$

where $x \in \mathbf{X} \subset \mathbb{R}^n$, $z \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^u$, $\gamma : \mathbb{R}^n \to \mathbb{R}^s$ and $\sigma : \mathbb{R}^r \times \mathbb{R}^p \to \mathbb{R}^q$. The matrices $A(x(t))$, $B(x(t))$, $C(x(t))$ and $G(x(t))$ are continuous in x on the compact set $\mathbf{X}$. We now consider the neural mass model used in [25] which was taken from [57]. Such model has the form of (6.30) with the following matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -a^2 & -2a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -a^2 & -2a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -b^2 & -2b \end{bmatrix}, \quad G(x(\varepsilon t)) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ x_1(t)ac_2 & 0 \\ 0 & 0 \\ 0 & x_2(t)bc_4 \end{bmatrix},$$

$$B(x(t)) = \begin{bmatrix} 0 & x_1(t)a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1(t)a & 0 & 0 \end{bmatrix}^T, \quad H = \begin{bmatrix} c_1 & 0 & 0 & 0 & 0 & 0 \\ c_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and $C = [0\ 0\ 1\ 0\ -1\ 0]$, where the parameters a, b, c_1, c_2, c_3 and c_4 are assumed to be known. The nonlinear terms in (6.30) are $\gamma = (S, S)$ and $\sigma(u, y) = (S(y), u)$ where the function S denotes the sigmoid function $S(v) := \frac{2e_0}{1+\exp[r(v_0, v)]}$ for $v \in \mathbb{R}$ with known constants e_0, v_0 and $r \in \mathbb{R}_{\geq 0}$. The states z_1, z_3 and z_5 are the membrane potential contributions of the pyramidal neurons, the excitatory and the inhibitory inter-neurons respectively, and z_2, z_4 and z_6 are their respective time derivatives. The unknown parameters x_1 and x_2 represent the synaptic gains of excitatory and inhibitory neuronal populations, respectively (see [57] for further details).

In Section 6.3.3, we considered the above single neural mass model with slowly time-varying parameters that belongs to $\mathbf{X} := [4, 8] \times [22, 28]$ and moves according to (6.31). Furthermore, we designed a multi-observer given by (6.32) and presented simulation results for it in Figures 6.2, 6.3 and 6.4. We demonstrated that the dynamic sampling policy introduced in [25] is not able to achieve arbitrarily small estimation errors when implemented on systems with unknown slowly-time varying parameters. In Table 7.1, we display the simulation parameters for the neural mass model we used in Section 6.3.3. We also use this parameters in the rest of this chapter.

$a = 100$	$b = 50$	$c_1 = 135$	$c_2 = 108$	$c_3 = 33.75$
$c_4 = 33.75$	$e_0 = 2.5$	$v_0 = 6$	$r = 0.56$	

Table 7.1: Simulation parameters for the neural mass model.

7.3 Simulation results for the new dynamic sampling policy

In this section, we present simulation results when we implement the new dynamic sampling policy introduced in Chapter 6. However, before presenting those simulation results, we briefly discuss the performance of the static sampling policy for parameter and state estimation of nonlinear systems with unknown slowly time-varying parameters. Consider the neural mass model presented above with parameters that slowly change according to (6.31).

We have found via simulations that the static sampling policy requires a large number of sample points and observers. When using the static sampling policy, a sampled set with 100 sample points generates a parameter estimation error with the following norm $|e_{x\sigma(t)}| = 0.62$ for $t > 180$. A sampled parameter set with 250 sample points generates a parameter estimation error with $|e_{x\sigma(t)}| = 0.18$ for $t > 160$. To generate a parameter estimation error with norm $|e_{x\sigma(t)}| = 0.05$, we need to generate 400 sample points, and subsequently, 400 observers.

We now present and analyse simulation results for the single neuron model when we implement the new dynamic sampling policy introduced in Chapter 6. We present the parameter estimation errors in Figure 7.1. Observe that after a finite time these estimation errors converge to a neighbourhood around zero. Our simulations were performed by using 100 sample points and they lead to a parameter estimation error ultimately bounded by 0.003 for $t \geq 200$.

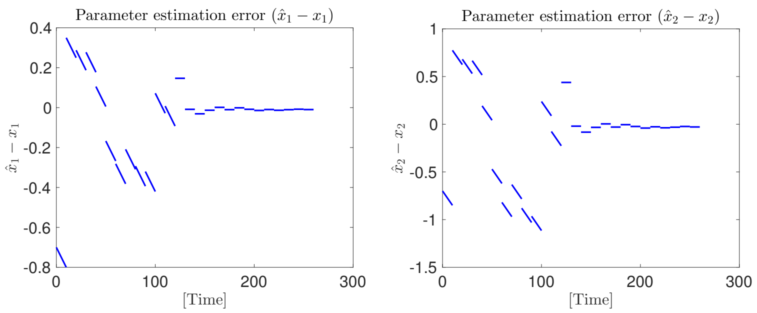

Figure 7.1: Parameter estimation errors for the neural mass model when using the multi-observer approach with the dynamic sampling policy introduced in Chapter 6.

In Figure 7.2, we illustrate how the parameter sampled set can recapture the real parameter when it has left the set. Note that during the first 10 iterations the multi-observer exhibits the same performance as simulation results in Section 6.3.3 where we used the approach from [25]. Hence, we only present iterations 1, 2, 9 and 10. The zoom-out procedure is executed at iteration eleven (plot in the centre of Figure 7.2). The rest of the plots correspond to iterations 12, 13, 19 and 25. Note that, as the parameter is recaptured by the sampled set, the parameter estimation error becomes arbitrarily small. This is the distinctive feature of our proposed dynamic sampling policy since the existing results in [25] cannot guarantee arbitrarily small errors as depicted by Figures 6.3 and 6.4.

The parameter estimates are presented in Figure 7.3 and the state estimation errors corresponding to the states z_3 and z_5 are displayed in Figure 7.4. We can compare plots in Figure 7.4 with those in Figure 6.4. In this case, the increasing performance of the state estimation errors is reverted once the parameter is recaptured by the sampled set.

7.4 Constant parameters and noisy measurements

We now consider nonlinear systems with unknown constant parameters and noisy measurements. We present simulation results for the cases of having vanishing and non-

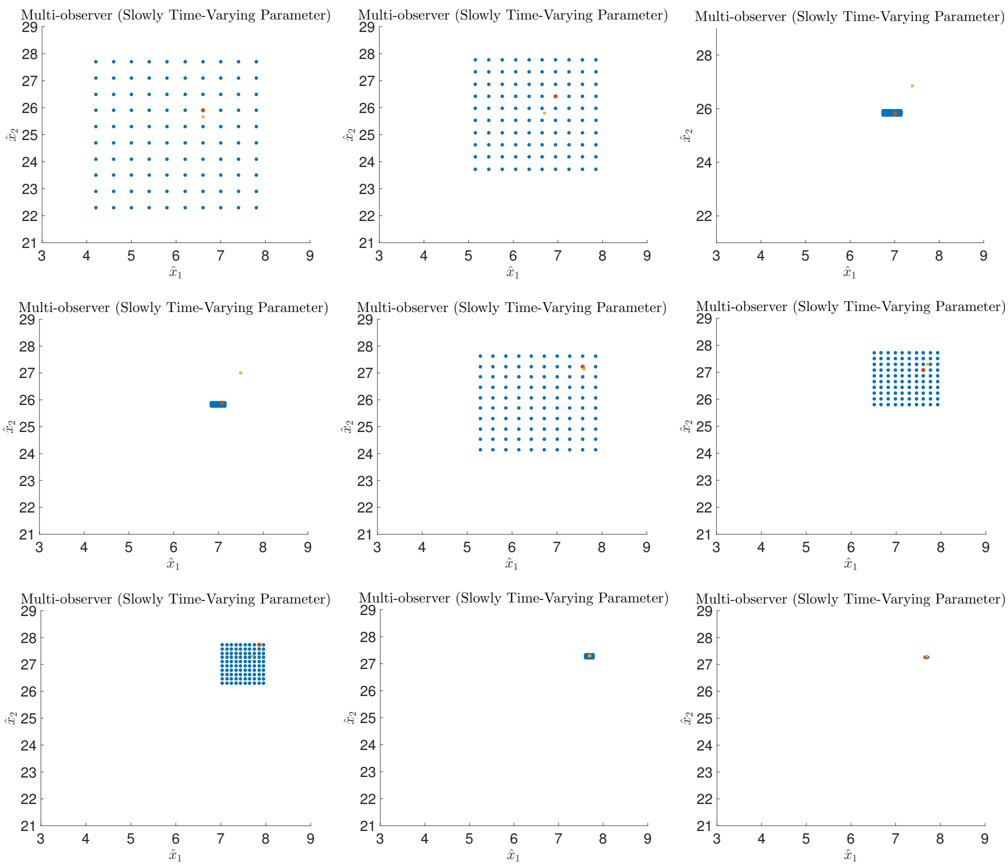

Figure 7.2: Yellow: Real parameter. Blue: Parameter sample points. Red: Parameter estimate.

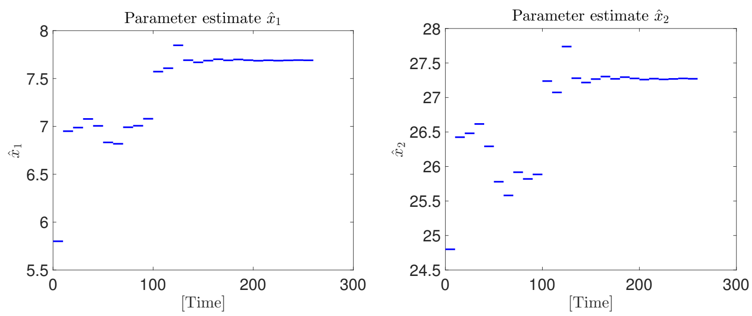

Figure 7.3: Parameter estimates when using the sampling policy introduced in Chapter 6.

vanishing noise. Here, we demonstrate via simulations that our proposed dynamic sampling policy in Chapter 6 can be used for parameter and state estimation of this class of systems. We consider the neural mass model presented in Section 7.2 with pa-

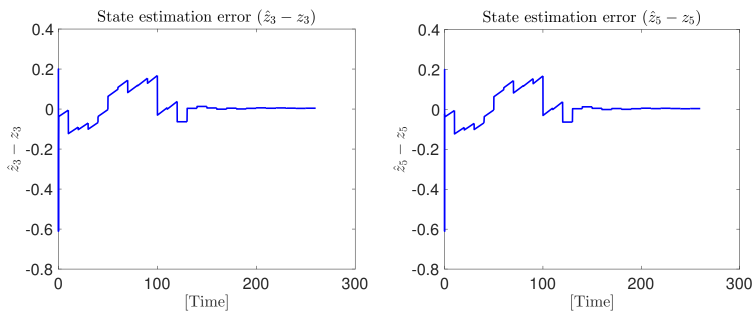

Figure 7.4: State estimation errors for z_3 and z_5 when using the new dynamic sampling policy introduced in Chapter 6.

rameters given by $x_1 = 6.5$ and $x_2 = 25.5$. Furthermore, we consider the following output for simulation purposes

$$y = \begin{cases} x_3 - x_5 + 0.08\sin(0.5t) - 0.01\cos(0.3t), & \text{if } t \leq 70, \\ x_3 - x_5, & \text{otherwise.} \end{cases} \tag{7.2}$$

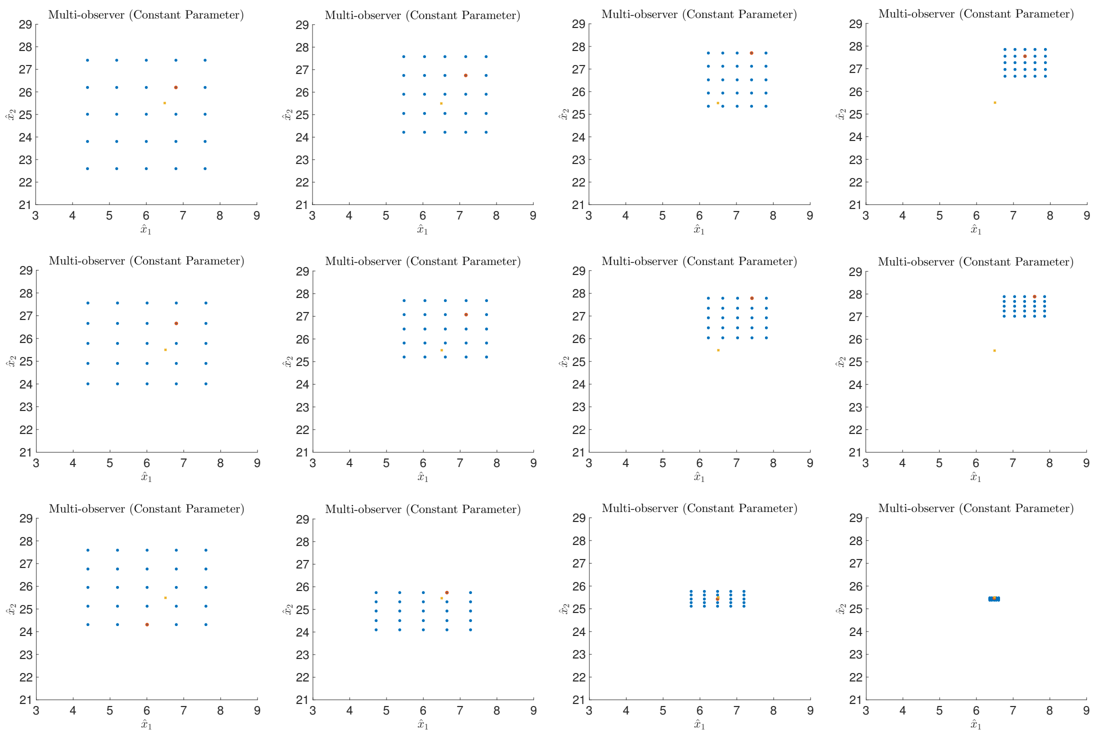

Figure 7.5: Simulation results for systems with unknown constant parameters and noisy measurements. Yellow: Real parameter. Blue: Parameter sample points. Red: Parameter estimate.

The simulation results in Figure 7.5 illustrate that the multi-observer approach uses the zoom-out procedure to allow to the estimates to follow the real parameter. We show the first ten iterations as the zoom-out procedure is executed at iterations four and eight. Since the measurement noise vanishes after the seventh iteration, the multi-observer can recover the parameter within the parameter sampled set and generate accurate estimates. This is illustrated by the last two plots in Figure 7.5.

We now present the norm of the parameter and state estimation errors obtained via simulations for the case when the noise does not vanish. These values are displayed in Tables 7.2, 7.3 and 7.4 for different values of the noise. Simulation results show that when the noise does not vanish, the estimation errors cannot be reduced arbitrarily. However, the estimation errors exhibit input-to-state stability-like behaviours with gain from noise which is a desired property when dealing with noisy measurements.

	$N = 25$	$N = 81$	$N = 100$
$\lvert\hat{x} - x\rvert_\infty$	0.367	0.242	0.091
$\lvert\hat{z} - z\rvert_\infty$	0.052	0.052	0.045

Table 7.2: ∞-norm of parameter and statee estimation errors when the noise satisfies $|w|_\infty = 0.05$.

	$N = 25$	$N = 81$	$N = 100$
$\lvert\hat{x} - x\rvert_\infty$	1.15	0.755	0.321
$\lvert\hat{z} - z\rvert_\infty$	0.155	0.101	0.093

Table 7.3: ∞-norm of parameter and statee estimation errors when the noise satisfies $|w|_\infty = 0.1$.

	$N = 25$	$N = 81$	$N = 100$
$\lvert\hat{x} - x\rvert_\infty$	1.58	0.764	0.378
$\lvert\hat{z} - z\rvert_\infty$	0.206	0.114	0.104

Table 7.4: ∞-norm of parameter and statee estimation errors when the noise satisfies $|w|_\infty = 0.15$.

7.5 Time-varying discontinuous parameters

We now show simulation results that illustrate that the multi-observer approach in Chapter 6 can be used on systems with discontinuous parameters with a sufficiently large dwell time (see Remark 7.1 below). Here, we perform simulations for the neural mass model when the parameter evolves in time satisfying the following

$$x_1 = \begin{cases} 6.5 + 0.01t & \text{if } t < 60, \\ 6 & \text{otherwise,} \end{cases} \tag{7.3a}$$

$$x_2 = \begin{cases} 25.5 + 0.015t & \text{if } t < 60 \\ 26.5 & \text{otherwise,} \end{cases} \tag{7.3b}$$

so that at $t = 60$ the parameter jumps to another value and stays there forever. We present simulation results in Figure 7.6. Note that the proposed dynamic sampling policy in Chapter 6 can follow the real parameter even after it has jumped.

Remark 7.1. *The discontinuous slowly time-varying parameter must have a sufficiently large* τ_d *dwell-time such that there is a* τ_d *time interval between consecutive discontinuities. This is a needed condition for our approach to work as monitoring signals require of a sufficiently large time to be able to deliver useful information about the estimates.*

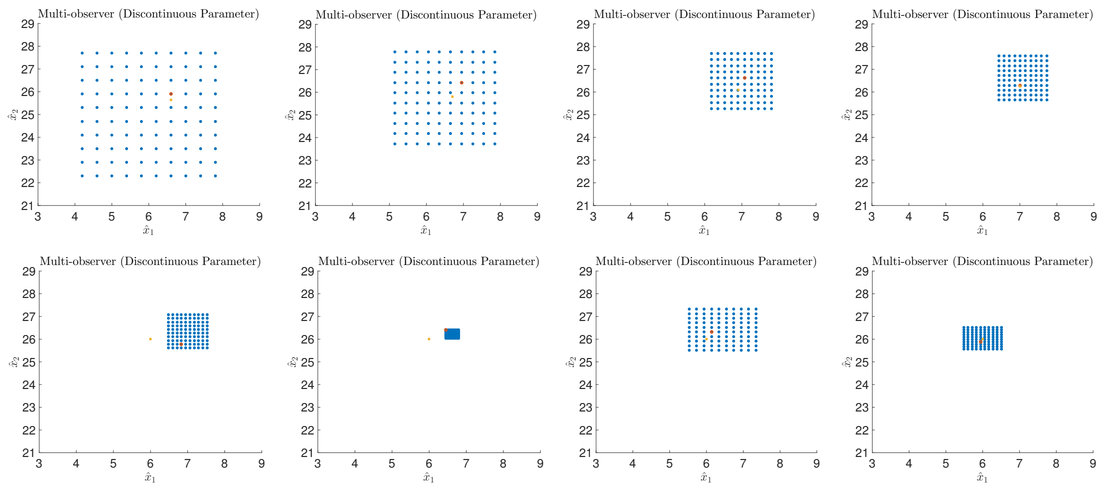

Figure 7.6: Simulation results for systems with unknown discontinuous parameters. Yellow: Real parameter. Blue: Parameter sample points. Red: Parameter estimate.

In Figure 7.6, we first display the first four iterations. Then, in the first column of the second row we present the sixth iteration where the parameter jumps from its continuous trajectory to the point $x = (6, 26.5)$. The zoom-out procedure is triggered at the tenth iteration which is showed in the third column of the second raw. This tracking of the real parameter is only possible because of the zoom-out procedure. Hence, results from [25] cannot address this problem.

7.6 A singularly perturbed plant

In this section, we present simulation results for the case when we apply the multi-observer approach and the new dynamic sampling policy from Chapter 6 to a singularly perturbed system. We consider a modified version of the suspension system used in Section 5.3.2. Here, we assume that the system has a linear spring element between the car body and the tire. Hence, the model of the system is given by

$$\dot{x}_1 = x_2 - z_2, \tag{7.4a}$$

$$\dot{x}_2 = -x_1 - \beta(x_2 - z_2) + u, \tag{7.4b}$$

$$\varepsilon \dot{z}_1 = z_2, \tag{7.4c}$$

$$\varepsilon \dot{z}_2 = \alpha x_1 - \alpha\beta(z_2 - x_2) - z_1 - \alpha u, \tag{7.4d}$$

where $\alpha = 2.28$, $\beta = 0.099$ and $\varepsilon = 0.0091$. We assume that an inertial measurement unit, located on the spring connecting the car body and the tire, is used to measure the spring deflection; such a sensor provides an output in the form of

$$y = z_1. \tag{7.5}$$

To implement the multi-observer technique, we only consider the fast dynamics (7.4c) - (7.4d) and treat the slow states as slowly time-varying parameters without using their model. Observe that a linear observer can be used to estimate the fast states of the system (7.4). Hence, let us consider the multi-observer

$$\dot{\hat{z}}_i = A(\hat{x}_i)\hat{z}_i + B(\hat{x}_i)u + L(\hat{x}_i)(C(\hat{x}_i)\hat{z}_i - y), \tag{7.6a}$$

$$\hat{y}_i = C(\hat{x}_i)\hat{z}_i, \tag{7.6b}$$

for $i \in \{1, \dots, N\}$, where $L(\hat{x}_i)$ is the observer gain which satisfies that $A(\hat{x}_i) + L(\hat{x}_i)C(\hat{x}_i)$ is Hurwitz (this is always possible whenever the pair $(A(\hat{x}_i), C(\hat{x}_i))$ is detectable). Note that $\hat{x}_i \in \mathbf{X}$ are the samples of the slow state and $\mathbf{X}$ is a compact set. Simulations results are displayed in Figure 7.7 where we have shown how the sampled set (set of blue points) moves around the compact set **X**. Observe that the dynamic sampling policy switches between zoom-in and zoom-out to be able to follow the real slow state. We have drawn a black 'box' to indicate an approximation of the set where the slow states evolve.

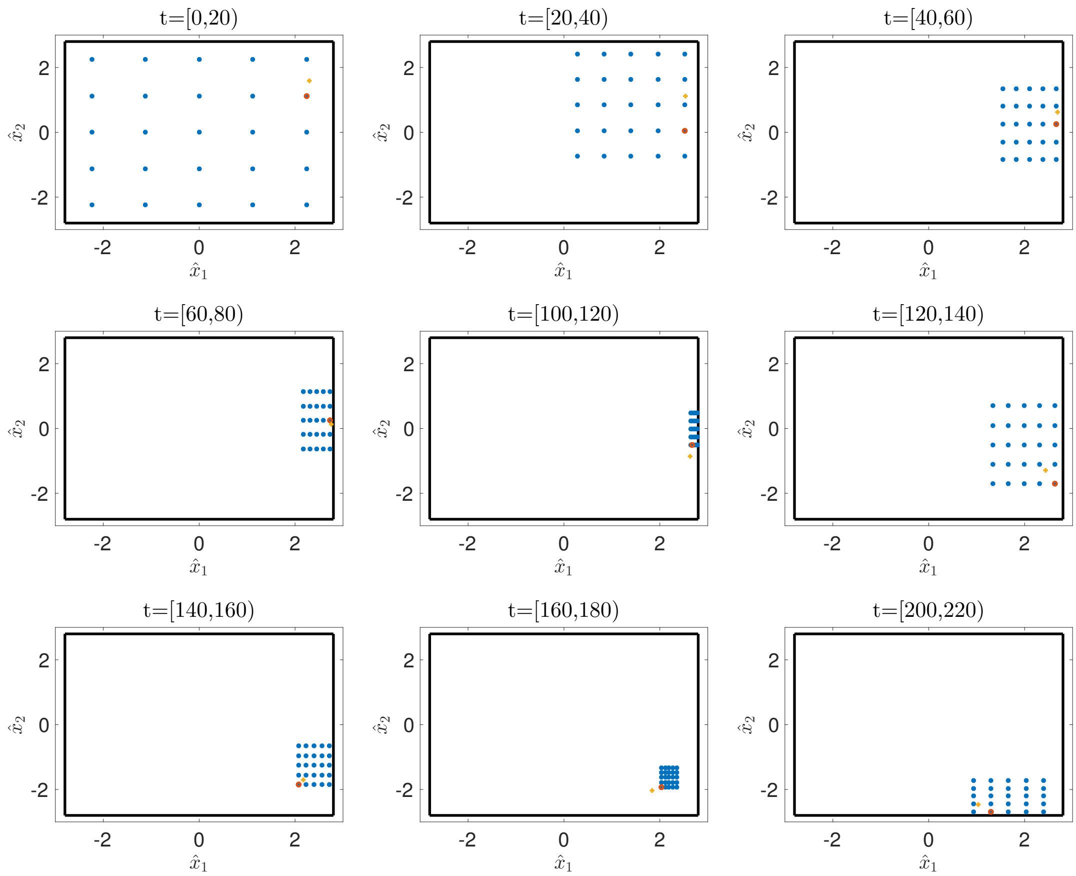

Figure 7.7: Simulation results for the simplified suspension system (7.4) - singularly perturbed plant. Yellow: Real parameter. Blue: Parameter sample points. Red: Parameter estimate.

Remark 7.2. *Simulation results showed that the new dynamic sampling policy introduced in Chapter 6 is capable of dealing with the full-state estimation of singularly perturbed systems. However, further improvements are required to reduce the complexity of tuning the multi-observer's parameters as well as for enhancing its performance. Here, the model of the slow dynamics has been ignored as the slow states have been treated as slowly varying parameters. Hence, a more efficient multi-observer approach can be generated by considering the structure of the slow part of the model.*

7.7 Conclusions of the chapter

We illustrated via simulations that the dynamic sampling policy described by Algorithm 6.2 performs well when used for parameter and state estimation of nonlinear systems with unknown slowly time-varying parameters. We demonstrated that our new dynamic sampling policy leads to arbitrarily small parameter and state estimation errors which cannot be addressed by results in [25]. We then showed that the new dynamic sampling policy can deal with other problems as the parameter estimation of systems with noisy measurements. We presented simulations for the case when the noise vanishes after a finite time. Moreover, we summarised in tables the norm of the estimation errors for the case when the measurement noise does not vanish. These results suggested that the estimation errors have an ultimate bound depending on the norm of the noise. We also presented simulation results that illustrate that results in Chapter 6 can be used on systems with discontinuous parameters with an appropriate τ_d dwell-time between discontinuities. Finally, we demonstrated via simulations that Algorithm 6.2 can be used on singularly perturbed systems to solve the full-state estimation problem. A rigorous mathematical study for these cases is left for future work.

Chapter 8
Conclusions and Future Work

8.1 Summary of Contributions

THIS THESIS has addressed the estimation problem of nonlinear singularly perturbed systems. We used standard singular perturbations techniques to study the slow state estimation of nonlinear plants with two time-scales. Although we used the standard approach, we addressed an estimation problem with a generality that has not been considered in the literature before. The generality of our results is implied by the fact that they cover a large class of nonlinear plants and observers of general dimension. We have also proposed a novel dynamic sampling policy for the multi-observer approach under the supervisory framework to address the parameter and state estimation of systems with unknown slowly time-varying parameters. These results are natural to the singular perturbations framework as the slow state can be regarded as slowly time-varying parameter to the fast part of the system. We now summarise the contributions of this thesis.

In Part I (Chapters 2 and Chapter 3), we generated a general design framework for the estimation of slow state of globally Lipschitz singularly perturbed systems. We performed a robustness analysis with respect to singular perturbations and to measurement noise for nonlinear full-order observers for globally Lipschitz nonlinear systems designed based on the reduced system to estimate the slow states of a singularly perturbed plant. The main contribution of this part of the thesis is the generality of our results since our assumptions hold for many nonlinear systems and observers as it was demonstrated in Chapter 3. Furthermore, our work also distinguishes from other results in the literature by the fact that we dealt with singularly perturbed systems with outputs corrupted by measurement noise. Although results in Part I cover smaller classes of systems and observers than results in Part II, our findings in Part I are important contributions since they allow us to obtain sharper conclusions for some observers than those generated by results in Part II.

In Part II (Chapter 4 and Chapter 5), we further generalised results in Part I by stating semi-global results under relaxed assumptions at the expense of concluding weaker convergence properties. These results cover boarder classes of nonlinear singularly perturbed systems and observers of general dimensions. Observe that both set of results in Parts I and II are important by their own right as they can be useful in different situations. For instance, when using a globally Lipschitz nonlinear observer, results in Part II only can guarantee semi-global convergence of the estimation error while results of Part I lead to sharper conclusions that hold globally. Hence, both Chapters 2 and 4 contain results that do not imply each other. Similarly to Part I, we used a standard singular perturbations approach to address the estimation problem and deliver a general estimation framework that applies to many nonlinear systems and observers. An important feature of results in Part II is that they cover reduced-order, full-order and higher-order observers. We stated practical input-to-state stability results as well as $\mathcal{L}_\infty \cap \mathcal{L}_2$ results when the measurement noise belongs to $\mathcal{L}_\infty \cap \mathcal{L}_2$. Furthermore, we concluded semi-global practical asymptotical stability of the estimation error in the absence of measurement noise as a direct consequence of our main results. We demonstrated and illustrated the applicability of our results by showing how several nonlinear systems and observers satisfy our assumptions and by presenting simulation results for numerical examples.

In Part III (Chapter 6 and Chapter 7), we introduced a multi-observer technique for parameter and state estimation of nonlinear systems with slowly time-varying parameters under the supervisory framework. We proposed a novel dynamic sampling policy for the multi-observer approach that is able to deal with plants with unknown parameters that are slowly changing. We rigorously proved in Chapter 6 that the multi-observer approach provides parameter and state estimates that are uniformly ultimately bounded if the unknown parameter moves sufficiently slowly. Furthermore, we illustrated in Chapter 7 that our proposed sampling policy addresses other problems as parameter and state estimation of systems with unknown constant parameters and noisy measurements, and the estimation problem of systems with unknown discontinuous and slowly time-varying parameters. The multi-observer technique is natural to the singular perturbations framework as the slow state can be regarded as slowly time-varying parameter to the fast part of the system.

8.2 Future Work

The results presented in this thesis open new directions for further research on the estimation problem of nonlinear singularly perturbed systems. In Part I and II of this thesis, we assumed that the algebraic equation defining the slow manifold has an isolated solution that can be obtained analytically. This assumption allowed us to obtain well defined reduced order and boundary layer systems. Hence, a plausible problem formulation for future work would be the slow state estimation problem when the reduced system is constructed via an approximation of the solution to the algebraic equation defining the slow manifold. This relaxed assumption would lead to an interesting problem where the errors due to the approximation will affect the estimation error properties. Then, an extensive study is required to characterise the stability of the slow estimation error under these approximation errors.

Future work can be done by considering nonlinear singularly perturbed systems with boundary layer solutions that do not necessarily converge to an equilibrium but they converge to a bounded set. For instance, the case when the trajectories of the boundary layer system converge to a family of limit cycles parametrized by the slow state. This scenario can be studied by using results on averaging methods in [118] where the steady-state performance of the boundary layer system can be used to average the derivative of the slow state. Then, a reduced averaged system approximates the behaviour of the slow part of the system and it can be used to estimate the slow variables. Hence, future opportunities on slow state estimation by using the reduced averaged system would lead to results addressing more general problems that are not covered by this thesis.

Here, we generated an estimation framework that allows to use a number of existing nonlinear observers for the slow state estimation of singularly perturbed systems. More constructive approaches to deal with this problem would lead to new research direction. Hence, a different perspective is the development of nonlinear observers exclusively dedicated for the slow state estimation of nonlinear singularly perturbed systems. We next focus on future research opportunities regarding parameter and state estimation problem of systems with slowly time-varying parameters and their implications on full state estimation of nonlinear singularly perturbed systems.

We illustrated in Chapter 7 that the new dynamic sampling policy introduced in Chapter 6 is useful to address different problems as the parameter and state estimation of nonlinear systems with constant unknown parameters and noisy measurements.

We also showed via simulations that the new sampling policy can address the estimation problem of systems with unknown discontinuous slowly time-varying parameters. Hence, an extensive and rigorous analysis is required to mathematically justify the convergence properties of the multi-observer approach in these situations.

As demonstrated in Section 7.6, the estimation technique introduced in Chapter 6 is natural to the singular perturbations framework as the slow states can be regarded as slowly time-varying parameters to the fast dynamics of the system. However, further research is needed to improve the multi-observer technique when used for full state estimation of singularly perturbed systems. When dealing with systems with two time-scales, we can generate a more efficient multi-observer approach by considering the structure of the reduced part of the model. This would help to reduce the required number of observers, and subsequently, the required computational power.

Another future work direction is depicted by Figure 8.1. This estimation technique would potentially produce accurate estimates of the full state after the transient has terminated. The methodology has three modes of operation that would be applied on three time intervals as shown in Figure 8.1. This approach would lead to a switched estimator as a result of combining a multi-observer for systems with slowly time-varying parameters, a slow observer synthesized for the reduced model and a fast estimator with slowly varying gains. Observe that this estimation technique would be highly useful in those cases where we need to preserve the time scale separation in the estimation error dynamics.

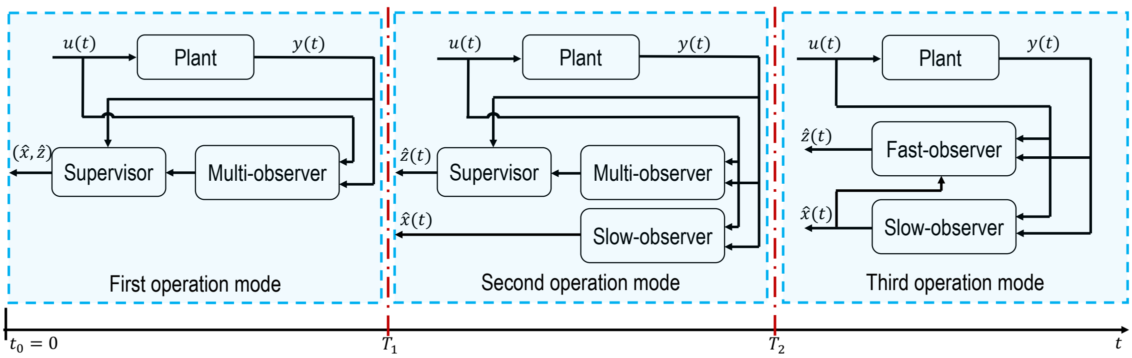

Figure 8.1: Schematic representation of the proposed estimation approach.

We now briefly describe the operation modes in Figure 8.1. During the first operation mode, we use a multi-observer to estimate the fast and slow state of the system in which the slow state is regarded as slowly time-varying parameter. This technique would deliver quickly estimates of the full state by requiring high computational power.

In the second mode, we start using the dynamics of the reduced system to estimate the slow variable while running a multi-observer to obtain the fast estimates. This mode is initialised once the estimation error of the slow state obtained from the first mode is sufficiently small so that the transient of the slow observer is reduced. Even though the multi-observer could potentially obtain very accurate slow estimates using only the first mode, the required computational power would make it impractical for large systems as nonlinear networks. Hence, using the reduced order model would improve the accuracy of the estimates without significantly increasing the computational cost.

The third mode starts once the error of the slow estimate is further reduced so that the effect of the initial conditions of the slow observer is "forgotten"; this would guarantee that in the third mode the slow estimation error is sufficiently and uniformly small for all time. As a result, we can replace the multi-observer with a single fast observer whose gains depend directly on the slow state estimate obtained from the slow observer. Since the slow estimate in the third mode would be sufficiently and uniformly close to the true slow state, both slow and fast observers would deliver estimates that ultimately produce arbitrarily small estimation errors. The most important feature of using this mode is that replacing the multi-observer by the slow and fast observers would significantly reduce the computational cost while obtaining accurate fast and slow estimates.

This technique would offer the flexibility to deal with the trade-offs of computational requirements, domain of attraction, speed of convergence and the ultimate bound on errors. We consider that the results from this thesis and further modifications of results in Chapter 6 can lead to the generation of the hybrid observer depicted in Figure 8.1. As far as we are aware, the estimation problem within the linear/nonlinear singular perturbations framework has never been addressed from this perspective. Hence, the proposed approach in Figure 8.1 is an interesting future research direction arising from the work presented in this thesis.

Here, we stated our results under semi-global conditions. Hence, as we usually do not know the initial conditions of real systems, a method to rationally choose the initial conditions of the observer is crucial when applying our results. Note that this is not an issue for our global results in Chapter 2 when the interconnection conditions are verified as stated in Remarks 2.2 and 2.5. However, the generation of systematic methods for appropriately choosing the initial conditions of the observers is an interesting area for further research in the semi-global case. This would contribute to ensuring the applicability of our results, and it would enrich the nonlinear observer design literature.

Appendix A

Proofs of Chapter 2

A.1 Proof of Lemma 2.1

We prove the result in two steps. In step 1) we show that the practical DISS condition in (2.19) holds. Then, in step 2), we prove that the practical $\mathcal{L}_2$ bound in (2.20) holds too. We use a linear version of the practical $\mathcal{L}_2$ stability definition used in [Property I_3, 95].

Step 1) Let Assumptions 2.1 - 2.6 hold. Define

$$\tilde{\varepsilon}^* := \frac{b_3 c_3}{2\left(\hat{b}_4 \hat{b}_1 + b_3 \hat{b}_6 + \frac{c_3}{2}\hat{b}_5\right)}, \tag{A.1}$$

where $\hat{b}_1 = c_4(L_3 + L_4 + L_0 L_5 + L_2 L_5) + c_6(L_0 + L_2)$, $\hat{b}_4 = b_5(L_0 + L_1)$, $\hat{b}_5 = b_5 L_0 + 1/4$ and $\hat{b}_6 = \hat{b}_2 + \hat{b}_3 + 1/2$, with $\hat{b}_3 = c_4 L_0 L_5 + c_6 L_0$ and $\hat{b}_2 = c_4(L_3 + L_1 L_5) + c_5 + c_6 L_1$. Moreover, define

$$k_1 := \sqrt{\frac{(1-\eta^*)b_2 + \eta^* c_2}{\min\{(1-\eta^*)b_1, \eta^* c_1\}}}, \qquad k_2 := \frac{\hat{\lambda}}{4((1-\eta^*)b_2 + \eta^* c_2)}, \tag{A.2}$$

$$k_3 := \sqrt{2k_1[(1-\eta^*)b_4 + \eta^* \hat{b}_1^2]\hat{\lambda}^{-1}}, \qquad k_4 := \sqrt{2k_1(1-\eta^*)(b_5 L_0)^2 \hat{\lambda}^{-1}}, \tag{A.3}$$

$$k_5 := \sqrt{2k_1 \eta^{*2} \hat{b}_3^2 \hat{\lambda}^{-1}}, \qquad k_6 := \sqrt{2k_1 \eta^* \frac{L_6^2}{2c_3}\hat{\lambda}^{-1}}, \tag{A.4}$$

$$k_7 := \sqrt{2k_1 \delta_{V_1}(1-\eta^*)\hat{\lambda}^{-1}}, \tag{A.5}$$

where $\eta^* = \frac{\hat{b}_4}{\hat{b}_4 + \hat{b}_1}$, and $\hat{\lambda} = \lambda_{\min}\{\mathbf{A}(\tilde{\varepsilon}^*, \eta^*)\}$ with

$$\mathbf{A}(\tilde{\varepsilon}^*, \eta^*) = \begin{bmatrix} (1-\eta^*)(b_3 - \tilde{\varepsilon}^* \hat{b}_5) & -\frac{1}{2}[(1-\eta^*)\hat{b}_4 + \eta^* \hat{b}_1] \\ -\frac{1}{2}[(1-\eta^*)\hat{b}_4 + \eta^* \hat{b}_1] & \frac{\eta}{\tilde{\varepsilon}^*}\left[\frac{c_3}{2} - \tilde{\varepsilon}^* \hat{b}_6\right] \end{bmatrix}. \tag{A.6}$$

All of the above constants come from Assumptions 2.3 - 2.6. We now consider the composite Lyapunov function

$$V(t,x,\xi) = (1-\eta)V_1(t,x) + \eta W(t,x,\xi), \tag{A.7}$$

where $\eta \in (0,1)$ is a constant to be chosen. The candidate Lyapunov functions $V_1(t,x)$ and $W(t,x,\xi)$ come from Assumptions 2.3 and 2.4 respectively. We take the time derivative of $V(t,x,\xi)$ along the solutions of (2.7). Then, we have that

$$\begin{aligned}\dot{V}\big|_{(2.7)} = (1-\eta)\left[\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x}f_s\right] + \eta\left[\frac{\partial W}{\partial t} + \frac{\partial W}{\partial x}f_s + \frac{1}{\varepsilon}\frac{\partial W}{\partial \xi}f_f\right.\\ \left.-\frac{\partial W}{\partial \xi}\left(\frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}f_s + \frac{\partial H}{\partial u}\dot{u}\right)\right].\end{aligned} \tag{A.8}$$

Now, we add and subtract terms to (A.8) that leads to

$$\begin{aligned}\dot{V}\big|_{(2.7)} = (1-\eta)\left[\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x}\overline{f_s} + \frac{\partial V_1}{\partial x}[f_s - \hat{f_s}] + \frac{\partial V_1}{\partial x}[\hat{f_s} - \overline{f_s}]\right] + \eta\left[\frac{1}{\varepsilon}\frac{\partial W}{\partial \xi}\overline{f_f}\right.\\ \left.+\frac{1}{\varepsilon}\frac{\partial W}{\partial \xi}[f_f - \overline{f_f}] + \frac{\partial W}{\partial t} + \frac{\partial W}{\partial x}f_s - \frac{\partial W}{\partial \xi}\left(\frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}f_s + \frac{\partial H}{\partial u}\dot{u}\right)\right],\end{aligned} \tag{A.9}$$

where $\hat{f_s}$, $\overline{f_s}$ and $\overline{f_f}$ denote $f_s(t,x,\xi{+}H,u,0)$, $f_s(t,x,H,u,0)$ and $f_f(t,x,\xi{+}H,u,0)$, respectively. By taking the norm over some terms on the right-hand side of (A.9), we obtain

$$\begin{aligned}\dot{V}\big|_{(2.7)} \le (1-\eta)\left[\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x}\overline{f_s} + \left|\frac{\partial V_1}{\partial x}\right|\left|f_s - \hat{f_s}\right| + \left|\frac{\partial V_1}{\partial x}\right|\left|\hat{f_s} - \overline{f_s}\right|\right] + \eta\left[\frac{1}{\varepsilon}\left|\frac{\partial W}{\partial \xi}\right||f_f - \overline{f_f}|\right.\\ \left.+\frac{1}{\varepsilon}\frac{\partial W}{\partial \xi}\overline{f_f} + \left|\frac{\partial W}{\partial t}\right| + \left|\frac{\partial W}{\partial x}\right||f_s| - \left|\frac{\partial W}{\partial \xi}\right|\left(\left|\frac{\partial H}{\partial t}\right| + \left|\frac{\partial H}{\partial x}\right||f_s| + \left|\frac{\partial H}{\partial u}\right||\dot{u}|\right)\right].\end{aligned} \tag{A.10}$$

We now use all the conditions defined in Assumptions 2.3 - 2.6 to obtain the following

$$\begin{aligned}\dot{V}\big|_{(2.7)} \le\ & (1-\eta)\left(-b_3|x|^2 + b_4|u|^2 + \varepsilon b_5 L_0(|x| + |\xi| + |u|)|x| + b_5 L_1|x||\xi| + \delta_{V_1}\right)\\ &+\eta\left(\frac{1}{\varepsilon}\varepsilon c_4 L_3(|x| + |\xi| + |u|)|\xi| - \frac{1}{\varepsilon}c_3|\xi|^2 + c_5|\xi|^2 + c_6\left(\varepsilon L_0(|x| + |\xi| + |u|)\right.\right.\\ &\left.+L_1|\xi| + L_2(|x| + |u|)\right)|\xi| + c_4|\xi|\left[L_4(|x| + |u|) + L_5\left(\varepsilon L_0(|x| + |\xi| + |u|)\right.\right.\\ &\left.\left.\left.+L_1|\xi| + L_2(|x| + |u|)\right) + L_6|\dot{u}|\right]\right).\end{aligned} \tag{A.11}$$

Observe that equation (A.11) leads to

$$\dot{V}\big|_{(2.7)} \leq -\begin{bmatrix}|x|^2\\|\xi|^2\end{bmatrix}^{T}\tilde{\mathbf{A}}(\varepsilon,\eta)\begin{bmatrix}|x|^2\\|\xi|^2\end{bmatrix} + \varepsilon(1-\eta)b_5L_0|u||x| + \varepsilon\eta\hat{b}_3|u||\xi| + \eta L_6|\dot{u}||\xi|$$
$$+\eta\hat{b}_1|u||\xi| + b_4(1-\eta)|u|^2 + (1-\eta)\delta_{V_1}, \tag{A.12}$$

with

$$\tilde{\mathbf{A}}(\varepsilon,\eta) = \begin{bmatrix}(1-\eta)(b_3-\varepsilon b_5L_0) & -\frac{1}{2}[(1-\eta)\hat{b}_4+\eta\hat{b}_1]\\ -\frac{1}{2}[(1-\eta)\hat{b}_4+\eta\hat{b}_1] & \frac{\eta}{\varepsilon}\left[c_3-\varepsilon(\hat{b}_2+\hat{b}_3)\right]\end{bmatrix},$$

where $\hat{b}_1 = c_4(L_3+L_4+L_0L_5+L_2L_5)+c_6(L_0+L_2)$, $\hat{b}_2 = c_4(L_3+L_1L_5)+c_5+c_6L_1$, $\hat{b}_3 = c_4L_0L_5+c_6L_0$ and $\hat{b}_4 = b_5(L_0+L_1)$. Note that completion of squares implies that $\eta L_6|\dot{u}||\xi| \leq \frac{\eta c_3}{2\varepsilon}|\xi|^2+\frac{\varepsilon\eta L_6^2}{2c_3}|\dot{u}|^2$, then by applying completion of squares to other cross terms too, we obtain from (A.12) that

$$\dot{V}\big|_{(2.7)} \leq -\begin{bmatrix}|x|^2\\|\xi|^2\end{bmatrix}^{T}\mathbf{A}(\varepsilon,\eta)\begin{bmatrix}|x|^2\\|\xi|^2\end{bmatrix} + \varepsilon(1-\eta)(b_5L_0)^2|u|^2 + \eta\varepsilon^2\hat{b}_3^2|u|^2 + \varepsilon\eta\frac{L_6^2}{2c_3}|\dot{u}|^2$$
$$+\eta\hat{b}_1^2|u|^2 + b_4(1-\eta)|u|^2 + (1-\eta)\delta_{V_1}, \tag{A.13}$$

with

$$\mathbf{A}(\varepsilon,\eta) = \begin{bmatrix}(1-\eta)(b_3-\varepsilon\hat{b}_5) & -\frac{1}{2}[(1-\eta)\hat{b}_4+\eta\hat{b}_1]\\ -\frac{1}{2}[(1-\eta)\hat{b}_4+\eta\hat{b}_1] & \frac{\eta}{\varepsilon}\left[\frac{c_3}{2}-\varepsilon\hat{b}_6\right]\end{bmatrix}, \tag{A.14}$$

where $\hat{b}_5 = b_5L_0 + 1/4$ and $\hat{b}_6 = \hat{b}_2+\hat{b}_3+1/2$. By the Sylvester's Criterion, the matrix $\mathbf{A}(\varepsilon,\eta)$ is positive definite if

$$(1-\eta)(b_3-\varepsilon\hat{b}_5) \geq 0, \tag{A.15}$$
$$(1-\eta)(b_3-\varepsilon\hat{b}_5)\frac{\eta}{\varepsilon}\left[\frac{c_3}{2}-\varepsilon\hat{b}_6\right]\frac{1}{4}[(1-\eta)\hat{b}_4+\eta\hat{b}_1]^2 \geq 0. \tag{A.16}$$

Note that both conditions, (A.15) and (A.16), hold if (A.16) holds. The inequality (A.16) leads to

$$\varepsilon < \frac{b_3c_3}{2\left\{\frac{1}{4\eta(1-\eta)}((1-\eta)\hat{b}_4+\eta\hat{b}_1)^2 + b_3\hat{b}_6 + \frac{c_3}{2}\hat{b}_5\right\}} := \varepsilon_\eta.$$

It is seen that the maximum value of ε_η occurs at $\eta = \eta^*$ with $\eta^* := \frac{\hat{b}_4}{\hat{b}_4+\hat{b}_1}$, and is given by (A.1). Therefore, for the fixed $\eta = \eta^*$, the matrix $\mathbf{A}(\varepsilon, \eta^*)$ is positive definite if $\varepsilon \in (0, \tilde{\varepsilon}^*)$. Now, we show that, for a fixed $\eta = \eta^*$, the quadratic part of (A.13) has an upper bound which is independent of ε.

Lemma A.1. *For the given matrix (A.14) and any $\eta \in (0, 1)$, the minimum eigenvalue $\lambda_{\min}\{(\mathbf{A}(\varepsilon, \eta)\}$ is strictly decreasing function of ε for every $\varepsilon > 0$.*

By virtue of Lemma A.1, $\lambda_{\min}\{\mathbf{A}(\varepsilon, \eta)\}$ is a strictly decreasing function of ε for $\eta = \eta^*$ and $\varepsilon > 0$. This fact implies that $\lambda_{\min}\{\mathbf{A}(\varepsilon, \eta^*)\} > \lambda_{\min}\{\mathbf{A}(\tilde{\varepsilon}^*, \eta^*)\}$ for all $\varepsilon \in (0, \tilde{\varepsilon}^*)$. Moreover, at $\eta = \eta^*$, it can be proven that there is an $\underline{\varepsilon}^* > \tilde{\varepsilon}^*$ at which the left-hand side of (A.16) is zero. Since $\tilde{\varepsilon}^* < \underline{\varepsilon}^*$, we conclude that $\lambda_{\min}\{\mathbf{A}(\tilde{\varepsilon}^*, \eta^*)\} > 0$. Then, it follows that

$$\begin{aligned} \dot{V}\big|_{(2.7)} \leq & -\lambda_{\min}\{\mathbf{A}(\tilde{\varepsilon}^*, \eta^*)\}|(x, \xi)|^2 + \bar{k}_3|u|^2 + \varepsilon\bar{k}_4|u|^2 \\ & + \varepsilon^2\bar{k}_5|u|^2 + \varepsilon\bar{k}_2|\dot{u}|^2 + (1-\eta^*)\delta_{V_1}, \end{aligned} \tag{A.17}$$

where $\mathbf{A}(\tilde{\varepsilon}^*, \eta^*)$ is given in (A.6), $\bar{k}_2 := \eta^* \frac{L_6^2}{2c_3}$, $\bar{k}_3 = (1-\eta^*)b_4 + \eta^*\hat{b}_1^2$, $\bar{k}_4 = (1-\eta^*)b_5^2L_0^2$, and $\bar{k}_5 = \eta^*\hat{b}_3^2$. Let use $\bar{k}_1 = \hat{\lambda}$ so that

$$\dot{V}\big|_{(2.7)} \leq -\bar{k}_1|(x, \xi)|^2 + \bar{k}_3|u|^2 + \varepsilon\bar{k}_4|u|^2 + \varepsilon^2\bar{k}_5|u|^2 + \varepsilon\bar{k}_2|\dot{u}|^2 + (1-\eta^*)\delta_{V_1}. \tag{A.18}$$

Assumptions 2.3 and 2.4 lead to

$$\bar{k}_6|(x, \xi)|^2 \leq V(t, x, \xi) \leq \bar{k}_7|(x, \xi)|^2, \tag{A.19}$$

where $\bar{k}_6 = \min\{(1-\eta^*)b_1, \eta^*c_1\}$ and $\bar{k}_7 = (1-\eta^*)b_2 + \eta^*c_2$. Then, it follows from (A.18) and (A.19) that

$$\begin{aligned} |(x(t), \xi(t))| \leq & \sqrt{\frac{\bar{k}_7}{\bar{k}_6}} \exp\left[-\frac{\bar{k}_1}{4\bar{k}_7}(t-t_0)\right] |(x_0, \xi_0)| + \sqrt{\frac{2}{\bar{k}_1}} \\ & \times \sqrt{(\bar{k}_3 + \varepsilon\bar{k}_4 + \varepsilon^2\bar{k}_5)|u[t_0, t]|^2 + \varepsilon\bar{k}_2|\dot{u}[t_0, t]|^2 + \bar{k}_8}, \end{aligned} \tag{A.20}$$

where $\bar{k}_8 = (1-\eta^*)\delta_{V_1}$. By using the inequality $\sqrt{a^2+b^2} \leq a + b$ for non-negative numbers a and b, it follows from (A.20) that the system (2.7) is practical DISS satisfying (2.19) with gains (A.2) - (A.5). Therefore, for any $\varepsilon \in (0, \tilde{\varepsilon}^*)$, (2.19) holds.

Step 2) Let Assumptions 2.1 - 2.6 hold. Define

$$\ell_1 := \sqrt{\frac{(1-\eta^*)b_2 + \eta^* c_2}{\hat{\lambda}}}, \qquad \ell_5 := \sqrt{\frac{\eta^* \hat{b}_3^2}{\hat{\lambda}}}, \tag{A.21}$$

$$\ell_2 := \sqrt{\frac{(1-\eta^*)b_4 + \eta^* \hat{b}_1^2}{\hat{\lambda}}}, \qquad \ell_5 := \sqrt{\frac{\eta^* \frac{L_6^2}{2c_3}}{\hat{\lambda}}}, \tag{A.22}$$

$$\ell_3 := \sqrt{\frac{(1-\eta^*)b_5^2 L_0^2}{\hat{\lambda}}}, \qquad \ell_6 := \sqrt{\frac{(1-\eta^*)\delta_{V_1}}{\hat{\lambda}}}, \tag{A.23}$$

where $\eta^* := \frac{\hat{b}_4}{\hat{b}_4 + \hat{b}_1}$ and $\hat{\lambda} := \lambda_{\min}\{\mathbf{A}(\tilde{\varepsilon}^*, \eta^*)\}$ with $\mathbf{A}(\tilde{\varepsilon}^*, \eta^*)$ given in (A.6). All the above constants are come from Assumptions 2.3 - 2.6. Now, consider the Lyapunov function in (A.7) and all the steps to get (A.18), which we rewrite here

$$\dot{V}\big|_{(2.7)} \le -\overline{k}_1 |(x,\xi)|^2 + \overline{k}_3 |u|^2 + \varepsilon \overline{k}_4 |u|^2 + \varepsilon^2 \overline{k}_5 |u|^2 + \varepsilon \overline{k}_2 |\dot{u}|^2 + (1-\eta^*)\delta_{V_1}, \tag{A.24}$$

where $\overline{k}_1 = \lambda_{\min}\{\mathbf{A}(\tilde{\varepsilon}^*, \eta^*)\}$, $\overline{k}_2 := \eta^* \frac{L_6^2}{2c_3}$, $\overline{k}_3 = (1-\eta^*)b_4 + \eta^* \hat{b}_1^2$, $\overline{k}_4 = (1-\eta^*)b_5^2 L_0^2$, and $\overline{k}_5 = \eta^* \hat{b}_3^2$. Integrating both sides of (A.24) yields

$$\begin{aligned} V(t, x(t), \xi(t)) - V(t_0, x_0, \xi_0) &\le -\overline{k}_1 \int_{t_0}^{t} |(x(\tau), \xi(\tau))|^2 d\tau + \overline{k}_3 \int_{t_0}^{t} |u(\tau)|^2 d\tau \\ &+ \varepsilon \overline{k}_4 \int_{t_0}^{t} |u(\tau)|^2 d\tau + \varepsilon^2 \overline{k}_5 \int_{t_0}^{t} |u(\tau)|^2 d\tau + \varepsilon \overline{k}_2 \int_{t_0}^{t} |\dot{u}(\tau)|^2 d\tau + (1-\eta^*)\delta_{V_1} \int_{t_0}^{t} d\tau, \end{aligned} \tag{A.25}$$

where $x(t), \xi(t)$ are the solutions to the system (2.7), and u and $\dot{u}$ are inputs to the system. By using the fact that $V(t,x,\xi) \ge 0$, we obtain

$$\begin{aligned} \int_{t_0}^{t} |(x(\tau), \xi(\tau))|^2 d\tau &\le \frac{\overline{k}_3}{\overline{k}_1} \int_{t_0}^{t} |u(\tau)|^2 d\tau + \varepsilon \frac{\overline{k}_4}{\overline{k}_1} \int_{t_0}^{t} |u(\tau)|^2 d\tau + \varepsilon^2 \frac{\overline{k}_5}{\overline{k}_1} \int_{t_0}^{t} |u(\tau)|^2 d\tau \\ &+ \varepsilon \frac{\overline{k}_2}{\overline{k}_1} \int_{t_0}^{t} |\dot{u}(\tau)|^2 d\tau + \frac{(1-\eta^*)\delta_{V_1}}{\overline{k}_1} \int_{t_0}^{t} d\tau + \frac{1}{\overline{k}_1} V(t_0, x_0, \xi_0). \end{aligned} \tag{A.26}$$

Taking the square roots and using the inequality $\sqrt{a^2+b^2} \le a + b$ for non-negative numbers a and b, we obtain

$$|(x(t), \xi(t))|_{\mathcal{L}_2} \le \left(\sqrt{\frac{\overline{k}_3}{\overline{k}_1}} + \sqrt{\varepsilon \frac{\overline{k}_4}{\overline{k}_1}} + \varepsilon \sqrt{\frac{\overline{k}_5}{\overline{k}_1}} \right) |u(t)|_{\mathcal{L}_2} + \sqrt{\frac{(1-\eta^*)\delta_{V_1}}{\overline{k}_1} t}$$

$$+\sqrt{\varepsilon\frac{\overline{k}_2}{\overline{k}_1}}|\dot{u}(t)|_{\mathcal{L}_2}+\sqrt{\frac{1}{\overline{k}_1}V(t_0,x_0,\xi_0)}. \tag{A.27}$$

It follows from (A.19) and (A.27) that

$$|(x(t),\xi(t))|_{\mathcal{L}_2}\leq\sqrt{\frac{\overline{k}_7}{\overline{k}_1}}|(x_0,\xi_0)|+\left(\sqrt{\frac{\overline{k}_3}{\overline{k}_1}}+\sqrt{\varepsilon\frac{\overline{k}_4}{\overline{k}_1}}+\varepsilon\sqrt{\frac{\overline{k}_5}{\overline{k}_1}}\right)|u(t)|_{\mathcal{L}_2}$$
$$+\sqrt{\varepsilon\frac{\overline{k}_2}{\overline{k}_1}}|\dot{u}(t)|_{\mathcal{L}_2}+\sqrt{\frac{(1-\eta^*)\delta_{V_1}}{\overline{k}_1}t}. \tag{A.28}$$

Then, it follows from (A.28) that the system (2.7) is practical $\mathcal{L}_2$ stable and satisfies (2.20) with linear gains (A.21) - (A.23). Therefore, for any $\varepsilon\in(0,\tilde{\varepsilon}^*)$, (2.20) holds. This completes the proof. ■

A.2 Proof of Corollary 2.1

We split the proof in two steps. In step 1), we prove that the DISS property (2.21) holds. Then, in step 2), we show that the $\mathcal{L}_2$ bound holds.
Step 1) Let Assumptions 2.1 - 2.6 hold. Define

$$\overline{\varepsilon}^*=\min\left\{\tilde{\varepsilon}^*,\frac{\sqrt{d_2^2+2d_4}-d_2}{d_4}\right\}, \tag{A.29}$$

where $\tilde{\varepsilon}^*$ has been defined in (A.1), $d_2=c_4(L_3+L_1L_5)+c_5+c_6L_1$ and $d_4=c_6L_0+c_4L_0L_5$. Define

$$\tilde{k}_1:=\sqrt{\frac{c_2}{c_1}},\qquad \tilde{k}_2:=\frac{c_3}{8c_2}, \tag{A.30}$$
$$\tilde{k}_3:=\frac{4}{c_3}\left[c_4(L_3+L_4+L_2L_5)+c_6L_2\right]\tilde{k}_1,\qquad \tilde{k}_4:=\frac{4}{c_3}\left[c_6L_0+c_4L_0L_5\right]\tilde{k}_1, \tag{A.31}$$
$$\tilde{k}_5:=\frac{4c_4}{c_3}L_6\tilde{k}_1, \tag{A.32}$$

$d_1=c_4(L_3+L_4+L_2L_5)+c_6L_2$ and $d_3=c_4(L_3+L_4+L_2L_5)+c_6L_2$ where $A=\frac{12}{c_3}\sqrt{\frac{c_2}{c_1}}$. All the above constants come from Assumptions 2.4 - 2.6. We now consider the Lyapunov function $W(t,x,\xi)$ in Assumption 2.4 and take its derivative along the solutions of (2.7).

This leads to

$$\dot{W}\big|_{(2.7)} = \frac{\partial W}{\partial t} + \frac{\partial W}{\partial x} f_s + \frac{1}{\varepsilon}\frac{\partial W}{\partial \xi} f_f - \frac{\partial W}{\partial \xi}\left(\frac{\partial H}{\partial t} + \frac{\partial H}{\partial x} f_s + \frac{\partial H}{\partial u}\dot{u}\right). \tag{A.33}$$

Now, we add and subtract terms to (A.33) that leads to

$$\begin{aligned}\dot{W}\big|_{(2.7)} = {} & \frac{1}{\varepsilon}\frac{\partial W}{\partial \xi}\overline{f_f} + \frac{1}{\varepsilon}\frac{\partial W}{\partial \xi}\left[f_f - \overline{f_f}\right] + \frac{\partial W}{\partial t} + \frac{\partial W}{\partial x} f_s \\ & - \frac{\partial W}{\partial \xi}\left(\frac{\partial H}{\partial t} + \frac{\partial H}{\partial x} f_s + \frac{\partial H}{\partial u}\dot{u}\right),\end{aligned} \tag{A.34}$$

where $\overline{f_f}$ denotes $f_f(t, x, \xi + H, u, 0)$. By taking the norm on the right hand side of (A.34), we obtain

$$\begin{aligned}\dot{W}\big|_{(2.7)} = {} & \frac{1}{\varepsilon}\frac{\partial W}{\partial \xi}\overline{f_f} + \frac{1}{\varepsilon}\left|\frac{\partial W}{\partial \xi}\right| |f_f - \overline{f_f}| + \left|\frac{\partial W}{\partial t}\right| + \left|\frac{\partial W}{\partial x}\right| |f_s| \\ & - \left|\frac{\partial W}{\partial \xi}\right|\left(\left|\frac{\partial H}{\partial t}\right| + \left|\frac{\partial H}{\partial x}\right| |f_s| + \left|\frac{\partial H}{\partial u}\right| |\dot{u}|\right).\end{aligned} \tag{A.35}$$

We now use all the conditions defined in Assumptions 2.4 - 2.6. Note that Assumption 2.5 implies $|f_s| \leq \varepsilon L_0(|x| + |\xi| + |u|) + L_1|\xi| + L_2(|x| + |u|)$. Then, we have that

$$\begin{aligned}\dot{W}\big|_{(2.7)} \leq {} & -\frac{1}{\varepsilon}c_3|\xi|^2 + \frac{1}{\varepsilon}\varepsilon c_4 L_3(|x| + |\xi| + |u|)|\xi| + c_5|\xi|^2 + c_6\left(\varepsilon L_0(|x| + |\xi| + |u|)\right. \\ & \left. + L_1|\xi| + L_2(|x| + |u|)\right)|\xi| + c_4|\xi|[L_4(|x| + |u|) + L_5(\varepsilon L_0(|x| + |\xi| + |u|) + L_1|\xi| \\ & + L_2(|x| + |u|)) + L_6|\dot{u}|].\end{aligned} \tag{A.36}$$

It follows that (A.36) leads to

$$\begin{aligned}\dot{W}\big|_{(2.7)} \leq {} & -\frac{1}{\varepsilon}c_3|\xi|^2 + d_1|x||\xi| + d_2|\xi|^2 + d_3|u||\xi| + c_4L_6|\xi||\dot{u}| \\ & + \varepsilon d_4|x||\xi| + \varepsilon d_4|u||\xi| + \varepsilon d_4|\xi|^2,\end{aligned} \tag{A.37}$$

where $d_1 = c_4(L_3 + L_4 + L_2L_5) + c_6L_2$, $d_2 = c_4(L_3 + L_1L_5) + c_5 + c_6L_1$, $d_3 = c_4(L_3 + L_4 + L_2L_5) + c_6L_2$ and $d_4 = c_6L_0 + c_4L_0L_5$. Then, for any $\varepsilon \in (0, \overline{\varepsilon}_1^*)$ with $\overline{\varepsilon}_1^* = \frac{\sqrt{d_2^2 + 2d_4} - d_2}{d_4}$ which satisfies $\overline{\varepsilon}^* \leq \overline{\varepsilon}_1^*$, we have that

$$\dot{W}\big|_{(2.7)} \leq -\frac{c_3}{2\varepsilon}|\xi|^2 + d_1|x||\xi| + d_3|u||\xi| + c_4L_6|\xi||\dot{u}| + \varepsilon d_4|x||\xi| + \varepsilon d_4|u||\xi|. \tag{A.38}$$

Inequality (A.38) leads to

$$|\xi| \geq \frac{4\varepsilon}{c_3}\left[\, d_1|x| + d_3|u| + c_4L_6|\dot{u}| + \varepsilon d_4|x| + \varepsilon d_4|u|\right] \implies \dot{W}\big|_{(6.22)} \leq -\frac{c_3}{4\varepsilon}|\xi|^2. \tag{A.39}$$

Clearly, (A.39) implies that the $\xi - subsystem$ is ISS with respect to x, u and $\dot{u}$ with the following linear gains

$$\gamma_{x\to\xi} = \varepsilon\frac{4}{c_3}\sqrt{\frac{c_2}{c_1}}\left(\, d_1 + \varepsilon d_4\right), \tag{A.40}$$

$$\gamma_{u\to\xi} = \varepsilon\frac{4}{c_3}\sqrt{\frac{c_2}{c_1}}\left(\, d_3 + \varepsilon d_4\right), \tag{A.41}$$

$$\gamma_{\dot{u}\to\xi} = \varepsilon\frac{4}{c_3}\sqrt{\frac{c_2}{c_1}}c_4L_6. \tag{A.42}$$

Since we see $x(t)$ as input to the $\xi - subsystem$, we need this signal to be bounded in order to have a finite upper bound for $\xi(t)$. From (2.19) and using $|x(t)| \leq |(x(t), \xi(t))|$, we conclude that $x(t)$ is a bounded signal when treated as input to the $\xi - subsystem$ for all $\varepsilon \in (0, \tilde{\varepsilon}^*)$ where $\tilde{\varepsilon}^* \leq \bar{\varepsilon}^*$. Then, it follows from (A.39) that the solutions of the fast dynamics are DISS satisfying (2.21) with gains (A.30) - (A.32). Hence, for any $\varepsilon \in (0, \bar{\varepsilon}^*)$, (2.21) holds.

Step 2) Let Assumptions 2.1 - 2.6 hold. Define

$$\tilde{\ell}_1 := \frac{8c_2}{c_3}, \qquad \tilde{\ell}_2 := \frac{4\sqrt{2}d_1}{c_3}, \tag{A.43}$$

$$\tilde{\ell}_3 := \frac{4\sqrt{2}c_4L_6}{c_3}, \qquad \tilde{\ell}_4 := \frac{4\sqrt{2}d_4}{c_3}, \tag{A.44}$$

$$\tilde{\ell}_5 := \frac{4d_4}{c_3}, \tag{A.45}$$

where $d_1 = c_4(L_3 + L_4 + L_2L_5) + c_6L_2$. All the above constants come from Assumptions 2.4 - 2.6. We consider the Lyapunov function $W(t, x, \xi)$ in Assumption 2.4. It follows from above that the time derivative of $W(t, x, \xi)$ is upper bounded as showed in (A.38). Hence, we apply completion of squares to (A.38) so that we obtain

$$\begin{aligned}\dot{W}\big|_{(2.7)} \leq& -\frac{c_3}{2\varepsilon}|\xi|^2 + d_1\left(\frac{c_3}{16\varepsilon d_1}|\xi|^2 + \frac{4\varepsilon d_1}{c_3}|x|^2\right) + d_3\left(\frac{c_3}{16\varepsilon d_3}|\xi|^2 + \frac{4\varepsilon d_3}{c_3}|u|^2\right)\\ &+ c_4L_6\left(\frac{c_3}{16\varepsilon c_4L_6}|\xi|^2 + \frac{4\varepsilon c_4L_6}{c_3}|\dot{u}|^2\right) + \varepsilon d_4\left(\frac{c_3}{16\varepsilon^2 d_4}|\xi|^2 + \frac{4\varepsilon^2 d_4}{c_3}|x|^2\right)\\ &+ \varepsilon d_4\left(\frac{c_3}{8\varepsilon^2 d_4}|\xi|^2 + \frac{2\varepsilon^2 d_4}{c_3}|u|^2\right).\end{aligned} \tag{A.46}$$

Then, we have that (A.46) leads to

$$\dot{W}\big|_{(2.7)} \leq -\frac{c_3}{8\varepsilon}|\xi|^2 + \frac{4\varepsilon d_1^2}{c_3}|x|^2 + \frac{4\varepsilon d_3^2}{c_3}|u|^2 + \frac{4\varepsilon c_4^2 L_6^2}{c_3}|\dot{u}|^2 + \frac{4\varepsilon^3 d_4^2}{c_3}|x|^2 + \frac{2\varepsilon^3 d_4^2}{c_3}|u|^2. \quad \text{(A.47)}$$

We integrate both sides of (A.47) so that we obtain

$$\begin{aligned} W(t, x(t), \xi(t)) - W(t_0, x_0, \xi_0) \leq & -\frac{c_3}{8\varepsilon}\int_{t_0}^{t} |\xi(\tau)|^2 d\tau + \varepsilon\frac{4d_1^2}{c_3}\int_{t_0}^{t} |x(\tau)|^2 d\tau \\ & + \varepsilon\frac{4d_3^2}{c_3}\int_{t_0}^{t} |u(\tau)|^2 d\tau + \varepsilon\frac{4c_4^2 L_6^2}{c_3}\int_{t_0}^{t} |\dot{u}(\tau)|^2 d\tau \\ & + \varepsilon^3\frac{4d_4^2}{c_3}\int_{t_0}^{t} |x(\tau)|^2 d\tau + \varepsilon^3\frac{2d_4^2}{c_3}\int_{t_0}^{t} |u(\tau)|^2 d\tau, \end{aligned} \quad \text{(A.48)}$$

where $\xi(t)$ is the solution for the error dynamics and x, u and $\dot{u}$ are seeing as inputs to the $\xi - subsystem$. Since $W(t, x, \xi) \geq 0$, we have that

$$\begin{aligned} \int_{t_0}^{t} |\xi(\tau)|^2 d\tau \leq \; & \varepsilon^2\frac{32d_1^2}{c_3^2}\int_{t_0}^{t} |x(\tau)|^2 d\tau + \varepsilon^2\frac{32d_3^2}{c_3^2}\int_{t_0}^{t} |u(\tau)|^2 d\tau + \varepsilon^2\frac{32c_4^2 L_6^2}{c_3^2}\int_{t_0}^{t} |\dot{u}(\tau)|^2 d\tau \\ & + \varepsilon^4\frac{32d_4^2}{c_3^2}\int_{t_0}^{t} |x(\tau)|^2 d\tau + \varepsilon^4\frac{16d_4^2}{c_3^2}\int_{t_0}^{t} |u(\tau)|^2 d\tau + \varepsilon\frac{8}{c_3}W(t_0, x_0, \xi_0). \end{aligned} \quad \text{(A.49)}$$

Then, by taking the square roots in (A.49) and using the inequality $\sqrt{a^2 + b^2} \leq a + b$ for non-negative numbers a and b, we obtain

$$\begin{aligned} |\xi(t)|_{\mathcal{L}_2} \leq & \sqrt{\varepsilon\frac{8c_2}{c_3}}|\xi_0| + \varepsilon\frac{4\sqrt{2}d_1}{c_3}|x(t)|_{\mathcal{L}_2} + \varepsilon\frac{4\sqrt{2}d_3}{c_3}|u(t)|_{\mathcal{L}_2} + \varepsilon\frac{4\sqrt{2}c_4 L_6}{c_3}|\dot{u}(t)|_{\mathcal{L}_2} \\ & + \varepsilon^2\frac{4\sqrt{2}d_4}{c_3}|x(t)|_{\mathcal{L}_2} + \varepsilon^2\frac{4d_4}{c_3}|u(t)|_{\mathcal{L}_2}. \end{aligned} \quad \text{(A.50)}$$

Since we see $x(t)$ as input to the $\xi - subsystem$, we need this signal to be bounded in order to have a finite upper bound for $\xi(t)$. It follows from Lemma 2.1 that $x(t)$ is a bounded signal when treated as input to the $\xi - subsystem$ for all $\varepsilon \in (0, \tilde{\varepsilon}^*)$ where $\tilde{\varepsilon}^* \leq \bar{\varepsilon}^*$. Then, we conclude from (A.50) that the $\xi - subsystem$ (2.7b) is $\mathcal{L}_2$ stable and satisfies (2.22) with linear gains (A.43) - (A.45). Therefore, for any $\varepsilon \in (0, \tilde{\varepsilon}^*)$, (2.22) holds. This completes the proof. ■

A.3 Proof of Theorem 2.1

We prove the result in two steps. In step 1) we show that the practical ISS condition in (2.32) holds. Then, in step 2), we prove that the "practical" $\mathcal{L}_2$ bound in (2.33) holds too.
Step 1) Let Assumptions 2.1 - 2.8 hold. Define

$$\varepsilon^* := \tilde{\varepsilon}^*, \tag{A.51}$$

where $\tilde{\varepsilon}^*$ comes from Lemma 2.1 and let

$$\overline{k}_1 := \sqrt{a_2/a_1}, \qquad \overline{k}_2 := a_3/(4a_2), \tag{A.52}$$

$$\overline{k}_3 := 2a_5L_7\overline{k}_1/a_3, \qquad \overline{k}_4 := 2a_5L_0\overline{k}_1/a_3, \tag{A.53}$$

$$\overline{k}_5 := 2a_5(L_1+L_8)\overline{k}_1/a_3, \qquad \overline{k}_6 := 2a_4\overline{k}_1/a_3. \tag{A.54}$$

All the constants used from (A.52) to (A.54) come from Assumptions 2.3 - 2.8. We now consider the Lyapunov function $V_e(t,e)$ and take its derivative along solutions to (2.30), which is given by

$$\dot{V}_e\big|_{(2.30)} = \frac{\partial V_e}{\partial t} + \frac{\partial V}{\partial e} f_e(t,x,e,\xi+H,y,u,\varepsilon). \tag{A.55}$$

By adding and subtracting terms to (A.55), we obtain that

$$\dot{V}_e\big|_{(2.30)} = \frac{\partial V_e}{\partial t} + \frac{\partial V}{\partial e}\overline{f_e} + \frac{\partial V}{\partial e}\left[f_e - \overline{f_e}\right], \tag{A.56}$$

where $\overline{f_e}$ denotes $f_e(t,x,e,H,y_s,u,0)$. By using the definition of the error dynamics, we rewrite (A.56) as follows,

$$\dot{V}_e\big|_{(2.30)} = \frac{\partial V_e}{\partial t} + \frac{\partial V}{\partial e}\overline{f_e} + \frac{\partial V}{\partial e}\left[\, f_s - f_o - \overline{f_s} + \overline{f_o}\right], \tag{A.57}$$

where $\overline{f_o}$ and $\overline{f_s}$ denote $f_o(t,x-e,y_s,u)$ and $f_s(t,x,H,u,0)$, respectively. By using the norm and the set of inequalities in Assumptions 2.6, we have that

$$\dot{V}_e\big|_{(2.30)} = -a_3|e|^2 + a_4|w|^2 + a_5\,|e|\left|\, f_s - \overline{f_s}\right| + a_5\,|e|\left|\,\overline{f_o} - f_o\right|. \tag{A.58}$$

We now use Assumptions 2.5 and 2.8 that leads to

$$\dot{V}_e\big|_{(2.30)} = -a_3|e|^2 + a_4|w|^2 + \varepsilon a_5 L_0\,|e|\,(|x|+|\xi|+|u|)$$

$$+ \varepsilon a_5 L_7 |e| + a_5(L_1 + L_8)|e||\xi|, \tag{A.59}$$

which can be written as

$$\begin{aligned}\dot{V}_e\big|_{(2.30)} &= -\frac{1}{2}a_3|e|^2 - \frac{1}{2}a_3|e|^2 + \varepsilon a_5 L_0 |e| (|x| + |\xi| + |u|) + a_4|w|^2 \\ &\quad + \varepsilon a_5 L_7 |e| + a_5(L_1 + L_8)|e||\xi|.\end{aligned} \tag{A.60}$$

Then, we have that

$$|e| \geq \frac{2}{a_3}\left[\varepsilon m_1|x| + (\varepsilon m_1 + m_2)|\xi| + \varepsilon m_1|u| + \varepsilon m_3 + a_4|w|^2\right] \implies \dot{V}_e\big|_{(2.30)} = -\frac{1}{2}a_3|e|^2, \tag{A.61}$$

where $m_1 = a_5 L_0$, $m_2 = a_5(L_1 + L_8)$, and $m_3 = a_5 L_7$. Since the error dynamics are in cascade with the original state, we take x and ξ as inputs to the error dynamics. Then, (A.61) implies that the error dynamics are ISS with the following linear gains

$$\gamma_{x\to e} = \varepsilon\frac{2}{a_3}\sqrt{\frac{a_2}{a_1}}m_1, \qquad \gamma_{\xi\to e} = \varepsilon\frac{2}{a_3}\sqrt{\frac{a_2}{a_1}}m_1 + \frac{2}{a_3}\sqrt{\frac{a_2}{a_1}}m_2, \tag{A.62}$$

$$\gamma_{u\to e} = \varepsilon\frac{2}{a_3}\sqrt{\frac{a_2}{a_1}}m_1, \qquad \gamma_{w\to e} = \frac{2}{a_3}\sqrt{\frac{a_2}{a_1}}a_4. \tag{A.63}$$

Then, it follows that

$$\begin{aligned}|e(t)| \leq \sqrt{\frac{a_2}{a_1}}\exp\left[-\frac{a_3}{4a_2}(t - t_0)\right]|e_0| + \varepsilon\frac{2}{a_3}\sqrt{\frac{a_2}{a_1}}\Big[m_3 + m_1|x[t_0,t]| + m_1|\xi[t_0,t]| \\ + m_1|u[t_0,t]|\Big] + \frac{2}{a_3}\sqrt{\frac{a_2}{a_1}}m_2|\xi[t_0,t]| + \frac{2}{a_3}\sqrt{\frac{a_2}{a_1}}a_4|w[t_0,t]|.\end{aligned} \tag{A.64}$$

Note that the upper bound (A.64) is finite only if the states are bounded, i.e. Lemma 2.1 must hold. Then, our result holds if $\varepsilon \in (0, \tilde{\varepsilon}^*)$ where $\tilde{\varepsilon}^*$ comes from Lemma 2.1 and it is included in (A.51). It follows that $x(t)$ and $\xi(t)$ are bounded signals when considered as inputs to the error dynamics. Therefore, by using (A.52) - (A.54) it follows that (2.32) holds for all $\varepsilon \in (0, \varepsilon^*)$.

Step 2) Let Assumptions 2.1 - 2.2 hold. Define

$$\hat{\ell}_1 := \sqrt{\frac{4a_2}{a_3}}, \qquad \hat{\ell}_2 := \sqrt{\frac{4a_4}{a_3}}, \tag{A.65}$$

$$\hat{\ell}_3 := \frac{2\sqrt{2}a_5 L_0}{a_3}, \qquad \hat{\ell}_4 := \frac{a_5 L_7}{a_3}, \tag{A.66}$$

$$\hat{\ell}_5 := \frac{2\sqrt{2}a_5(L_1 + L_8)}{a_3}, \tag{A.67}$$

where all the constants come from Assumptions 2.3 - 2.8. Consider the Lyapunov function $V_e(t, e)$ and take its derivative along the solutions of (2.30), which is bounded as showed in (A.59). We now apply completion of squares to (A.59) to obtain the appropriate structure to conclude $\mathcal{L}_2$ stability results. Hence, we obtain

$$\begin{aligned}\dot{V}_e\big|_{(2.30)} \leq& -a_3|e|^2 + \frac{a_3}{4}|e|^4 + a_4|w|^2 + \varepsilon a_5 L_0\left(k_1|e|^2 + k_2|x|^2\right) + \varepsilon a_5 L_0\left(k_1|e|^2 + k_2|\xi|^2\right)\\ &+ \varepsilon a_5 L_0\left(k_1|e|^2 + k_2|u|^2\right) + \varepsilon^2\frac{a_5^2 L_7^2}{a_3} + a_5(L_1 + L_8)\left(k_3|e|^2 + k_4|\xi|^2\right).\end{aligned} \tag{A.68}$$

Let $k_1 = \frac{a_3}{8\varepsilon a_5 L_0}$, $k_2 = 2\frac{\varepsilon a_5 L_0}{a_3}$, $k_3 = \frac{a_3}{8[a_5(L_1+L_8)]}$ and $k_4 = \frac{2[a_5(L_1+L_8)]}{a_3}$. Then, we have that (A.68) leads to

$$\begin{aligned}\dot{V}_e\big|_{(2.30)} \leq& -\frac{a_3}{4}|e|^2 + a_4|w|^2 + 2\frac{(\varepsilon a_5 L_0)^2}{a_3}|x|^2 + 2\frac{(\varepsilon a_5 L_0)^2}{a_3}|\xi|^2\\ &+ 2\frac{(\varepsilon a_5 L_0)^2}{a_3}|u|^2 + 2\frac{a_5^2(L_1 + L_8)^2}{a_3}|\xi|^2 + \varepsilon^2\frac{a_5^2 L_7^2}{a_3}.\end{aligned} \tag{A.69}$$

We now integrate both sides of (A.69) so that

$$\begin{aligned}V_e(t, e(t)) - V_e(t_0, e_0) \leq& -\frac{a_3}{4}\int_0^t |e(\tau)|^2 d\tau + a_4\int_0^t |w(\tau)|^2 d\tau + 2\frac{a_5^2(L_1 + L_8)^2}{a_3}|\int_0^t \xi(\tau)|^2 d\tau\\ &+ 2\frac{(\varepsilon a_5 L_0)^2}{a_3}\int_0^t |x(\tau)|^2 d\tau + 2\frac{(\varepsilon a_5 L_0)^2}{a_3}\int_0^t |\xi(\tau)|^2 d\tau\\ &+ 2\frac{(\varepsilon a_5 L_0)^2}{a_3}\int_0^t |u(\tau)|^2 d\tau + \varepsilon^2\frac{a_5^2 L_7^2}{a_3}\int_0^t d\tau,\end{aligned} \tag{A.70}$$

where $e(\tau)$ is the solution for the error dynamics and x, ξ and u are seeing as inputs. Since $V_e(t, e) \geq 0$, we obtain

$$\begin{aligned}\int_0^t |e(\tau)|^2 d\tau \leq& \frac{4a_4}{a_3}\int_0^t |w(\tau)|^2 d\tau + 2\frac{4(\varepsilon a_5 L_0)^2}{a_3^2}\int_0^t |x(\tau)|^2 d\tau\\ &+ 2\frac{4(\varepsilon a_5 L_0)^2}{a_3^4}\int_0^t |\xi(\tau)|^2 d\tau + 2\frac{4a_5^2(L_1 + L_8)^2}{a_3^2}\int_0^t |\xi(\tau)|^2 d\tau\\ &+ 2\frac{4(\varepsilon a_5 L_0)^2}{a_3^4}\int_0^t |u(\tau)|^2 d\tau + \varepsilon^2\frac{a_5^2 L_7^2}{a_3^2}\int_0^t d\tau + \frac{4}{a_3}V_e(t_0, e_0).\end{aligned} \tag{A.71}$$

Then, taking the square roots in (A.71) and using the inequality $\sqrt{a^2+b^2} \leq a+b$ for non-negative numbers a and b, we obtain

$$
\begin{aligned}
|e(t)|_{\mathcal{L}_2} \leq \sqrt{\frac{4a_2}{a_3}}|e_0| + \sqrt{\frac{4a_4}{a_3}}|w(t)|_{\mathcal{L}_2} + \varepsilon\frac{2\sqrt{2}a_5L_0}{a_3}|x(t)|_{\mathcal{L}_2} + \varepsilon\frac{2\sqrt{2}a_5L_0}{a_3}|\xi(t)|_{\mathcal{L}_2} \\
+ \varepsilon\frac{2\sqrt{2}a_5L_0}{a_3}|u(t)|_{\mathcal{L}_2} + \varepsilon\frac{a_5L_7}{a_3}t + \frac{2\sqrt{2}a_5(L_1+L_8)}{a_3}|\xi(t)|_{\mathcal{L}_2}.
\end{aligned}
\tag{A.72}
$$

Note that the error dynamics is in cascade with (x, ξ). Since Assumptions 2.1 - 2.8 hold, Lemma 2.1 holds too. It follows from Lemma 2.1 that $x(t)$ and $\xi(t)$ are bounded; then, we can treat them as bounded input signals to the error dynamics. Hence, the $\mathcal{L}_2$ stability property implied by (A.72) holds if $\varepsilon \in (0, \tilde{\varepsilon}^*)$ where $\tilde{\varepsilon}^* \geq \varepsilon^*$ comes from Lemma 2.1. Therefore, it follows that (2.33) holds, with $\hat{\ell}_i$ for $i \in \{1, \ldots, 5\}$, given by (A.65) - (A.67), for all $\varepsilon \in (0, \varepsilon^*)$. This completes the proof.■

Appendix B

Proofs of Chapter 4

B.1 Proof of Lemma 4.1

We split the proof in three steps. In the first step, we prove that (4.15) and (4.16) hold under Assumptions 4.1 - 4.5. In the second step, we show that the full system is practical DISS, i.e., (4.17) holds. Finally, we prove in the last step that (4.6) satisfies the practical $\mathcal{L}_\infty \cap \mathcal{L}_2$ stability condition (4.18).

Step 1) Let Assumptions 4.1 - 4.5 hold. Define the composite Lyapunov function

$$V(t,x,\xi) := (1-d)V_1(t,x) + dW(t,x,\xi), \tag{B.1}$$

where $d \in (0,1)$ is to be chosen and $V_1(t,x)$ and $W(t,x,\xi)$ come from Assumptions 4.3 and 4.4, respectively. Moreover, let $d^* \in (0,1)$ be such that

$$d^* := \frac{b_1}{b_1 + b_2 + b_3}, \tag{B.2}$$

where $b_i \geq 0$, for $i \in \{1,2,3\}$ come from Assumption 4.5. Define

$$\underline{\alpha}_V(s) := \inf_{|(r_1,r_2)| \geq s} \{(1-d^*)\underline{\alpha}_{V_1}(r_1) + d^*\underline{\alpha}_W(r_2)\}, \tag{B.3a}$$

$$\overline{\alpha}_V(s) := (1-d^*)\overline{\alpha}_{V_1}(s) + d^*\overline{\alpha}_W(s), \tag{B.3b}$$

where $\underline{\alpha}_{V_1}(\cdot)$, $\overline{\alpha}_{V_1}(\cdot)$ come from Assumption 4.3 and $\underline{\alpha}_W(\cdot)$, $\overline{\alpha}_W(\cdot)$ come from Assumption 4.4. Define

$$\gamma_V(s) := (1-d^*)\gamma_{V_1}(s), \tag{B.4a}$$

$$\tilde{\gamma}_V(s) := (1-d^*)\gamma_1^2(s) + \frac{9d^*}{4\zeta_3}\left[\gamma_2^2(s) + \gamma_3^2(s)\right], \tag{B.4b}$$

$$\hat{\gamma}_V(s) := \frac{9d^*}{4\zeta_3}\gamma_4^2(s), \tag{B.4c}$$

$$\mu_V := (1-d^*)\delta_{V_1}, \tag{B.4d}$$

where $\gamma_{V_1}(\cdot)$, come from Assumption 4.3, $\gamma_i(\cdot)$, for $i \in \{1,\ldots,4\}$, come from Assumption 4.5, and ζ_3 comes from 4.4. Define

$$\tilde{\varepsilon}^* := \frac{\frac{2}{3}\zeta_1\zeta_3}{b_1(b_2+b_3)+\zeta_1(a_2+a_3)+\frac{2}{3}\zeta_3\left(a_1+\frac{1}{4}\right)}, \tag{B.5}$$

and

$$\alpha_V(s) := a_{l1} \inf_{|(r_1,r_2)|\geq s}\left\{\alpha_{V_1}^2(r_1)+\alpha_W^2(r_2)\right\}, \tag{B.6}$$

in which $a_{l1} := \lambda_{\min}\{\mathbf{A}(d^*,\tilde{\varepsilon}^*)\}$ where

$$\mathbf{A}(d^*,\tilde{\varepsilon}^*) := \begin{bmatrix} (1-d^*)\left[\zeta_1-\tilde{\varepsilon}^*\left(a_1+\frac{1}{4}\right)\right] & -\frac{1}{2}(1-d^*)b_1-\frac{1}{2}d^*(b_2+b_3) \\ -\frac{1}{2}(1-d^*)b_1-\frac{1}{2}d^*(b_2+b_3) & \frac{1}{\tilde{\varepsilon}^*}d^*\left(\frac{2}{3}\zeta_3-\tilde{\varepsilon}^*(a_2+a_3)\right)\end{bmatrix}, \tag{B.7}$$

where all constants and the class-$\mathcal{K}_\infty$ functions used in (B.5) - (B.7) come from Assumptions 4.3 - 4.5. We now consider (B.1) as candidate Lyapunov function for (4.6). By [Lemma 4.3, 70], there exist class-$\mathcal{K}_\infty$ functions $\underline{\alpha}_V(\cdot)$ and $\overline{\alpha}_V(\cdot)$ such that, for a fixed d given by (B.2), the Lyapunov function (B.1) satisfies (4.15) for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^m$, $u \in \mathbb{R}^r$, $\dot{u} \in \mathbb{R}^r$ and $t \geq 0$. Moreover, the functions $\underline{\alpha}_V(\cdot)$ and $\overline{\alpha}_V(\cdot)$ can be constructed from (4.4) and (4.10). It is straightforward to prove that the upper bound for the composite Lyapunov function (B.1) is given by (B.3b). To construct the lower bound, we consider the following result.

Lemma B.1. *Consider the functions* $\kappa_1(|r|), \kappa_2(|l|) \in \mathcal{K}_\infty$. *Then, the function*

$$\hat{\kappa}(s) = \inf_{|(r,l)|\geq s}\{\kappa_1(|r|)+\kappa_2(|l|)\}\,\forall\, s \geq 0, \tag{B.8}$$

is of class-$\mathcal{K}_\infty$ *function.*

The proof of Lemma B.1 is given in Appendix B.6. It follows from (4.4), (4.10) and Lemma B.1 that the lower bound for the composite Lyapunov function (B.1) is given by (B.3a) for all $s \geq 0$. Hence, (4.15) holds for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^m$, $u \in \mathbb{R}^r$, $\dot{u} \in \mathbb{R}^r$ and $t \geq 0$ with $\underline{\alpha}_V(s) \in \mathcal{K}_\infty$ and $\overline{\alpha}_V(s) \in \mathcal{K}_\infty$ given by (B.3a) and (B.3b), respectively. We now take the derivative of (B.1) along solutions of (4.6). So, we have

$$\dot{V}\big|_{(6.48)} = (1-d)\left(\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x} f_s\right) + d\left[\frac{\partial W}{\partial t} + \frac{\partial W}{\partial x} f_s + \frac{1}{\varepsilon}\frac{\partial W}{\partial \xi} f_f - \frac{\partial W}{\partial \xi}\left(\frac{\partial H}{\partial t} + \frac{\partial H}{\partial x} f_s + \frac{\partial H}{\partial u}\dot{u}\right)\right]. \tag{B.9}$$

By adding and subtracting terms to (B.9), we obtain

$$\dot{V}\big|_{(6.48)} = (1-d)\left(\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x} f_s + \frac{\partial V_1}{\partial x} f_{s_0} - \frac{\partial V_1}{\partial x} f_{s_0}\right) + d\left[\frac{\partial W}{\partial t} + \frac{\partial W}{\partial x} f_s + \frac{1}{\varepsilon}\frac{\partial W}{\partial \xi} f_f + \frac{1}{\varepsilon}\frac{\partial W}{\partial \xi} f_{f_0} - \frac{1}{\varepsilon}\frac{\partial W}{\partial \xi} f_{f_0} - \frac{\partial W}{\partial \xi}\left(\frac{\partial H}{\partial t} + \frac{\partial H}{\partial x} f_s + \frac{\partial H}{\partial u}\dot{u}\right)\right], \tag{B.10}$$

where $f_{s_0} = f_s(t, x, H(t, x, u), u, 0)$ and $f_{f_0}(t, x, \xi{+}H(t, x, \xi), u, 0)$. Using Assumptions 4.3 - 4.5, the time derivative of (B.1) is bounded as follows

$$\begin{aligned}\dot{V}\big|_{(6.48)} \leq &-\begin{bmatrix}\alpha_{V_1}(\cdot)\\ \alpha_W(\cdot)\end{bmatrix}^T \begin{bmatrix}(1-d)(\zeta_1 - \varepsilon a_1) & -\frac{1}{2}(1-d)b_1 - \frac{1}{2}d(b_2+b_3)\\ -\frac{1}{2}(1-d)b_1 - \frac{1}{2}d(b_2+b_3) & \frac{1}{\varepsilon}d\left(\zeta_3 - \varepsilon(a_2+a_3)\right)\end{bmatrix}\begin{bmatrix}\alpha_{V_1}(\cdot)\\ \alpha_W(\cdot)\end{bmatrix}\\ &+ (1-d)\delta_{V_1} + (1-d)\left[\gamma_{V_1}(|u|) + \varepsilon\gamma_1(|u|)\alpha_{V_1}(|x|)\right] + d[\gamma_2(|u|)\alpha_W(|\xi|)\\ &+ \gamma_3(|u|)\alpha_W(\xi) + \gamma_4(|\dot{u}|)\alpha_W(\xi)].\end{aligned} \tag{B.11}$$

We first prove that, for a fixed $d = d^*$ with d^* given in (B.2), the negative quadratic part in (B.11) is bounded by a function that depends on ε. Then, we show that this function has a bound independent of the perturbation parameter. So, we apply completion of squares to (B.11) to remove the cross terms $\gamma_1(|u|)\alpha_{V_1}(|x|)$, $\gamma_2(|u|)\alpha_W(|\xi|)$, $\gamma_3(|u|)\alpha_W(|\xi|)$ and $\gamma_4(|\dot{u}|)\alpha_W(|\xi|)$, which leads to

$$\begin{aligned}\dot{V}\big|_{(6.48)} \leq &-\begin{bmatrix}\alpha_{V_1}(\cdot)\\ \alpha_W(\cdot)\end{bmatrix}^T \begin{bmatrix}(1-d)(\zeta_1 - \varepsilon a_1) & -\frac{1}{2}(1-d)b_1 - \frac{1}{2}d(b_2+b_3)\\ -\frac{1}{2}(1-d)b_1 - \frac{1}{2}d(b_2+b_3) & \frac{1}{\varepsilon}d\left(\zeta_3 - \varepsilon(a_2+a_3)\right)\end{bmatrix}\begin{bmatrix}\alpha_{V_1}(\cdot)\\ \alpha_W(\cdot)\end{bmatrix}\\ &+ (1-d)\delta_{V_1} + (1-d)\gamma_{V_1}(|u|) + \varepsilon(1-d)[k_1\gamma_1^2(|u|) + k_2\alpha_{V_1}^2(|x|)] + d[k_3\gamma_2^2(|u|)\\ &+ k_4\alpha_W^2(|\xi|)] + d[k_3\gamma_3^2(|u|) + k_4\alpha_W^2(\xi)] + d[k_3\gamma_4^2(|\dot{u}|) + k_4\alpha_W^2(\xi)].\end{aligned} \tag{B.12}$$

Let define $k_1 := 1$, $k_2 := \frac{1}{4}$, $k_3 := \frac{9\varepsilon}{4\zeta_3}$, and $k_4 := \frac{\zeta_3}{9\varepsilon}$. So, we obtain form (B.12) that

$$\dot{V}\big|_{(6.48)} \leq -\begin{bmatrix}\alpha_{V_1}(\cdot)\\ \alpha_W(\cdot)\end{bmatrix}^T \begin{bmatrix}(1-d)\left[\zeta_1 - \varepsilon\left(a_1 + \frac{1}{4}\right)\right] & -\frac{1}{2}(1-d)b_1 - \frac{1}{2}d(b_2+b_3)\\ -\frac{1}{2}(1-d)b_1 - \frac{1}{2}d(b_2+b_3) & \frac{1}{\varepsilon}d\left(\frac{2}{3}\zeta_3 - \varepsilon(a_2+a_3)\right)\end{bmatrix}\begin{bmatrix}\alpha_{V_1}(\cdot)\\ \alpha_W(\cdot)\end{bmatrix}$$

$$+ (1-d)\delta_{V_1} + (1-d)\gamma_{V_1}(|u|) + \varepsilon(1-d)\gamma_1^2(|u|)$$
$$+ \varepsilon\frac{9d}{4\zeta_3}\left[\gamma_2^2(|u|) + \gamma_3^2(|u|) + \gamma_4^2(|\dot{u}|)\right]. \tag{B.13}$$

We rewrite (B.13) as follows

$$\dot{V}\big|_{(6.48)} \le -\mathbf{b}^{\mathsf{T}}\mathbf{A}(d,\varepsilon)\mathbf{b} + \gamma_{V_d}(|u|) + \varepsilon\tilde{\gamma}_{V_d}(|u|) + \varepsilon\hat{\gamma}_{V_d}(|\dot{u}|) + \mu_{d\delta}, \tag{B.14}$$

where $\mathbf{b} := [\alpha_{V_1}(\cdot), \alpha_W(\cdot)]^{\mathsf{T}}$, $\gamma_{V_d}(s) := (1-d)\gamma_{V_1}(s)$, $\hat{\gamma}_{V_d}(s) := \frac{9d}{4\zeta_3}\gamma_4^2(s)$, $\mu_{d\delta} := (1-d)\delta_{V_1}$, $\tilde{\gamma}_{V_d}(s) := (1-d)\gamma_1^2(s) + \frac{9d}{4\zeta_3}\left[\gamma_2^2(s) + \gamma_3^2(s)\right]$, and

$$\mathbf{A}(d,\varepsilon) := \begin{bmatrix} (1-d)\left[\zeta_1 - \varepsilon\left(a_1 + \frac{1}{4}\right)\right] & -\frac{1}{2}(1-d)b_1 - \frac{1}{2}d(b_2+b_3) \\ -\frac{1}{2}(1-d)b_1 - \frac{1}{2}d(b_2+b_3) & \frac{1}{\varepsilon}d\left(\frac{2}{3}\zeta_3 - \varepsilon(a_2+a_3)\right) \end{bmatrix}. \tag{B.15}$$

For the first term on the right-hand side of (B.14) to be negative, the square matrix $\mathbf{A}(d,\varepsilon)$ in (B.15) must be positive definite. By the Sylvester's Criterion, the matrix $\mathbf{A}(d,\varepsilon)$ is positive definite if

$$(1-d)\left[\zeta_1 - \varepsilon\left(a_1 + \frac{1}{4}\right)\right] \ge 0, \tag{B.16}$$
$$(1-d)\left[\zeta_1 - \varepsilon\left(a_1 + \frac{1}{4}\right)\right]\frac{d}{\varepsilon}\left[\frac{2}{3}\zeta_3 - \varepsilon(a_2+a_3)\right] - \frac{1}{4}\left[(1-d)b_1 + d(b_2+b_3)\right]^2 \ge 0. \tag{B.17}$$

Note that both conditions, (B.16) and (B.17), hold if (B.17) holds. It is observed that inequality (B.17) leads to

$$\varepsilon < \frac{\frac{2}{3}\zeta_1\zeta_3}{\frac{1}{4d(1-d)}\left[(1-d)b_1 + d(b_2+b_3)\right]^2 + \zeta_1(a_2+a_3) + \frac{2}{3}\zeta_3\left(a_1 + \frac{1}{4}\right)} := \varepsilon_d. \tag{B.18}$$

It is seen that the maximum value of ε_d is given by $\tilde{\varepsilon}^*$ in (B.5) and it occurs at $d = d^*$ with d^* given by (B.2). Therefore, for the fixed $d = d^*$, the matrix $\mathbf{A}(d^*,\varepsilon)$ is positive definite for all $\varepsilon \in (0, \tilde{\varepsilon}^*)$. We now show that, for a fixed $d = d^*$, the first term on the right-hand side of (B.14) has an upper bound which is independent of ε. To do so, consider the following result.

Lemma B.2. *For the given matrix (B.15) and any* $d \in (0,1)$*, the minimum eigenvalue* $\lambda_{\min}\{\mathbf{A}(d,\varepsilon)\}$ *is a strictly decreasing function of* ε *for every* $\varepsilon > 0$*.*

The proof for Lemma B.2 is given in Appendix B.7. By virtue of Lemma B.2, we conclude that $\lambda_{\min}\{\mathbf{A}(d^*, \varepsilon)\} > \lambda_{\min}\{\mathbf{A}(d^*, \tilde{\varepsilon}^*)\}$ for all $\varepsilon \in (0, \tilde{\varepsilon}^*)$. Moreover, at $d = d^*$, there is an $\underline{\varepsilon}^* > \tilde{\varepsilon}^*$ at which the left-hand side of (B.17) is zero (see proof of Lemma B.2). Then, we conclude that $\lambda_{\min}\{\mathbf{A}(d^*, \tilde{\varepsilon}^*)\} > 0$ (see Remark B.1). Then, it follows that the first term on the right-hand side of (B.14) has a bound which is independent of d and ε. Moreover, the input gains in (B.14) can be chosen independent of $d \in (0, 1)$ too. This leads to

$$\dot{V}\big|_{(6.48)} \leq -\lambda_{\min}\{\mathbf{A}(d^*, \tilde{\varepsilon}^*)\}\mathbf{b}^\mathsf{T}\mathbf{b} + \gamma_V(|u|) + \varepsilon\tilde{\gamma}_V(|u|) + \varepsilon\hat{\gamma}_V(|\dot{u}|) + \mu_V, \tag{B.19}$$

where $\gamma_V(\cdot), \tilde{\gamma}_V(\cdot), \hat{\gamma}_V(\cdot) \in \mathcal{K}_\infty$ and $\mu_V > 0$ are given by (B.4). Let define $\Theta(x, \xi) := \alpha^2_{V_1}(|x|) + \alpha^2_W(|\xi|)$, which is a radially unbounded function. So, by [Lemma 4.3, 70], there exists a class-$\mathcal{K}_\infty$ function $\Xi(|(x, \xi)|)$ such that $\Theta(x, \xi) \geq \Xi(|(x, \xi)|)$. Moreover, by virtue of Lemma B.1, $\Xi(|(x, \xi)|) := \inf_{|(x,\xi)|\geq s}\{\Theta(x, \xi)\}$. Hence, by defining $\alpha_V(\cdot) \in \mathcal{K}_\infty$ as in (B.6), we have that $-a_{l1}\mathbf{b}^\mathsf{T}\mathbf{b} \leq -\alpha_V(|(x, \xi)|)$ where $a_{l1} := \lambda_{\min}\{\mathbf{A}(d^*, \tilde{\varepsilon}^*)\}$ with $\mathbf{A}(d^*, \tilde{\varepsilon}^*)$ given by (B.7). Since a_{l1} is positive for a fixed value of d^* in (B.2) and $\tilde{\varepsilon}^*$ in (B.5), we have that $\alpha_V(\cdot)$ is in fact a function of class$-\mathcal{K}_\infty$. Therefore, (4.16) holds for all $\varepsilon \in (0, \tilde{\varepsilon}^*)$, and for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^m$, $u \in \mathbb{R}^r$, $\dot{u} \in \mathbb{R}^r$ and $t \geq 0$.

Step 2) We now show that (4.7) is practical DISS stable. Let $\lambda_{L_1}(\cdot, \cdot) \in \mathcal{KL}$ be defined as the solution of the following differential equation

$$\dot{y} = -\alpha_V \circ \overline{\alpha}_V^{-1}(y), \;\; y(t_0) = y_0,$$

where $\overline{\alpha}_V(\cdot)$ and $\alpha_V(\cdot)$ come from (4.15) and (4.16), respectively. Then, $y(t) = \lambda_{L_1}(y_0, t - t_0)$. The existence of $\lambda_{L_1}(\cdot, \cdot)$ follows from [Lemma 4.4, 70]. Define

$$\beta_{L_1}(r, s) := \underline{\alpha}_V^{-1}(\lambda_{L_1}(\overline{\alpha}_V(r), s)), \tag{B.20}$$

where $\underline{\alpha}_V(\cdot)$ and $\overline{\alpha}_V(\cdot)$ come from (4.15). Define

$$\gamma_{L_1}(s) := \underline{\alpha}_V^{-1} \circ \overline{\alpha}_V \circ \alpha_V^{-1}(4(1 - d^*)\gamma_{V_1}(s)), \tag{B.21a}$$

$$\tilde{\gamma}_\varepsilon(s) := \underline{\alpha}_V^{-1} \circ \overline{\alpha}_V \circ \alpha_V^{-1}\left(4\varepsilon(1 - d^*)\gamma_1^2(s) + \varepsilon\frac{9d^*}{\zeta_3}\left[\gamma_2^2(s) + \gamma_3^2(s)\right]\right), \tag{B.21b}$$

$$\hat{\gamma}_\varepsilon(s) := \underline{\alpha}_V^{-1} \circ \overline{\alpha}_V \circ \alpha_V^{-1}\left(\varepsilon\frac{9d^*}{\zeta_3}\gamma_4^2(s)\right), \tag{B.21c}$$

$$\mu_{L_1} := \underline{\alpha}_V^{-1} \circ \overline{\alpha}_V \circ \alpha_V^{-1}\left(4(1 - d^*)\delta_{V_1}\right), \tag{B.21d}$$

where the class-$\mathcal{K}_\infty$ functions $\underline{\alpha}_V(\cdot)$, $\overline{\alpha}_V(\cdot)$ and $\alpha_V(\cdot)$ come from (4.15) and (4.16), $\gamma_{V_1}(\cdot)$ and δ_{V_1} are given in Assumption 4.3, $\gamma_i(\cdot)$, for $i \in \{1, \ldots, 4\}$, come from Assumption 4.5, and ζ_3 is given in Assumption 4.4. We apply results in [114] and [116] to (4.15) and (4.16). So, it follows that the system (4.6) is practical DISS stable satisfying (4.17) with $\beta_{L_1}(\cdot,\cdot) \in \mathcal{KL}$ given by (B.20), $\gamma_{L_1}(\cdot), \tilde{\gamma}_\varepsilon(\cdot), \hat{\gamma}_\varepsilon(\cdot) \in \mathcal{K}_\infty$ defined as in (B.21a) - (B.21c), and $\mu_{L_1} > 0$ given by (B.21d). Therefore, (4.17) holds for all $\varepsilon \in (0, \tilde{\varepsilon}^*)$ and for all $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^m$, $u, \dot{u} \in \mathcal{L}_\infty$ and $t \geq t_0 \geq 0$.

Step 3) We now prove that the practical $\mathcal{L}_2$ stability condition (4.18) holds. To do so, we consider and integrate (4.16) as follows

$$\begin{aligned} V(t, x(t), \xi(t)) - V(t_0, x(t_0), \xi(t_0)) \leq -\int_{t_0}^{t} \alpha_V(|(x(\tau), \xi(\tau))|)d\tau + \int_{t_0}^{t} \gamma_V(|u(\tau)|)d\tau \\ +\varepsilon \int_{t_0}^{t} \tilde{\gamma}_V(|u(\tau)|)d\tau + \varepsilon \int_{t_0}^{t} \hat{\gamma}_V(|\dot{u}(\tau)|)d\tau + \mu_V \int_{t_0}^{t} d\tau, \quad \text{(B.22)} \end{aligned}$$

where $x(t)$ and $\xi(t)$ are the solutions of (4.6) and u and $\dot{u}$ are inputs to the system. We use the fact that $V(t, x(t), \xi(t)) \geq 0$ to obtain

$$\begin{aligned} \int_{t_0}^{t} \alpha_V(|(x(\tau), \xi(\tau))|)d\tau \leq V(t_0, x(t_0), \xi(t_0)) + \int_{t_0}^{t} \gamma_V(|u(\tau)|)d\tau + \varepsilon \int_{t_0}^{t} \tilde{\gamma}_V(|u(\tau)|)d\tau \\ + \varepsilon \int_{t_0}^{t} \hat{\gamma}_V(|\dot{u}(\tau)|)d\tau + \mu_V \int_{t_0}^{t} d\tau. \quad \text{(B.23)} \end{aligned}$$

It follows from (4.15) that $V(t_0, x(t_0), \xi(t_0)) \leq \overline{\alpha}_V(|(x_0, \xi_0)|)$. Then, (B.23) leads to (4.18). Therefore, (4.18) holds for all $\varepsilon \in (0, \tilde{\varepsilon}^*)$ and for all $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^m$, $u, \dot{u} \in \mathcal{L}_2$ and $t \geq t_0 \geq 0$. This completes the proof. ■

B.2 Proof of Corollary 4.1

We split the proof into three steps. In the first step, we prove the ultimate boundedness condition in (4.19). Then, we show that (4.20) holds in the second step. Finally, in the third step, we prove that the practical $\mathcal{L}_2$ the fast state satisfy the stability property in (4.21).

Step 1) Let Assumptions 4.1 - 4.5 hold. Let $\lambda_\xi(\cdot,\cdot) \in \mathcal{KL}$ be defined as the solution of the following scalar differential equation,

$$\dot{y}_\xi = -\frac{1}{\varepsilon}\hat{\alpha}_W(y_\xi), \;\; y_\xi(t_0) = y_{\xi_0},$$

with $\hat{\alpha}_W(\cdot) = \frac{\zeta_3}{4}\alpha_W^2 \circ \overline{\alpha}_W^{-1}(\cdot)$, where ζ_3, $\alpha_W(\cdot)$ and $\overline{\alpha}_W(\cdot)$ come from Assumption 4.4. Then, $y_\xi(t) = \lambda_\xi\left(y_{\xi_0}, \frac{t-t_0}{\varepsilon}\right)$. The existence of $\lambda_\xi(\cdot,\cdot)$ follows from [Lemma 4.4, 70]. Define the class-$\mathcal{KL}$ function

$$\beta_\xi(r,s) := \underline{\alpha}_W^{-1}\left(\lambda_\xi\left(\overline{\alpha}_W(r), s\right)\right), \tag{B.24}$$

where the functions $\underline{\alpha}_W(\cdot)$ and $\overline{\alpha}_W(\cdot)$ come from Assumption 4.4. Let $\tilde{\Delta} > 0$, $\tilde{\Delta}_{u_1} > 0$, $\tilde{\Delta}_{u_2} > 0$ and $\tilde{\mu} > 0$ be given such that $|(x_0, \xi_0)| \le \tilde{\Delta}$, $\|u\|_\infty \le \tilde{\Delta}_{u_1}$, and $\|\dot{u}\|_\infty \le \tilde{\Delta}_{u_2}$. By using Lemma 4.1, we generate $\tilde{\varepsilon}^* > 0$ such that (4.17) holds for all $\varepsilon \in (0, \tilde{\varepsilon}^*)$ and for all $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^m$, $u, \dot{u} \in \mathcal{L}_\infty$ and $t \ge t_0 \ge 0$. Introduce $\tilde{\Delta}_x := \beta_{L_1}(\tilde{\Delta}, 0) + \gamma_{L_1}(\tilde{\Delta}_{u_1}) + \tilde{\gamma}_\varepsilon(\tilde{\Delta}_{u_1}) + \hat{\gamma}_\varepsilon(\tilde{\Delta}_{u_2}) + \mu_{L_1}$ where $\beta_{L_1}(\cdot,\cdot)$, $\gamma_{L_1}(\cdot)$, $\tilde{\gamma}_\varepsilon(\cdot)$, $\hat{\gamma}_\varepsilon(\cdot)$ and μ_{L_1} come from Lemma 4.1. Then, $|x(t)| \le |(x(t), \xi(t))|$ and (4.17) imply that $|x(t)| \le \tilde{\Delta}_x$ for all $|(x_0, \xi_0)| \le \tilde{\Delta}$, $u \in \tilde{B}_{u_1}$, $\dot{u} \in \tilde{B}_{u_2}$ and $t \ge t_0 \ge 0$ where $\tilde{B}_{u_1} = \{u \in \mathbb{R}^r \mid |u| \le \tilde{\Delta}_{u_1}\}$ and $\tilde{B}_{u_2} = \{\dot{u} \in \mathbb{R}^r \mid |\dot{u}| \le \tilde{\Delta}_{u_2}\}$. Define

$$C := 4[(b_2 + b_3)^2\alpha_{V_1}^2(\tilde{\Delta}_x) + \gamma_2^2(\tilde{\Delta}_{u_1}) + \gamma_3^2(\tilde{\Delta}_{u_1}) + \gamma_4^2(\tilde{\Delta}_{u_2})], \tag{B.25}$$

and

$$\tilde{C} := \underline{\alpha}_W^{-1} \circ \overline{\alpha}_W \circ \tilde{\alpha}_W^{-1}\left(\varepsilon\frac{C}{\zeta_3}\right), \tag{B.26}$$

where $\tilde{\alpha}_W(\cdot) := \alpha_W^2(\cdot)$ with $\underline{\alpha}_W(\cdot)$, $\overline{\alpha}_W(\cdot)$, $\alpha_W(\cdot)$ come from Assumption 4.4, $\alpha_{V_1}(\cdot)$ is given by Assumption 4.3, $b_2 \ge 0$, $b_3 \ge 0$ and $\gamma_i(\cdot)$ $(i = 2, \ldots, 4)$ come from Assumption 4.5. Let $\overline{\varepsilon}^* > 0$ be such that $-\frac{1}{\varepsilon}\frac{\zeta_3}{2} + (a_2 + a_3 + 1) \le 0$. Note that we can reduce $\tilde{C}$ in (B.26) by reducing ε; so, for any given $\tilde{\mu}$ we can always obtain $\tilde{\mu} > \tilde{C}$ if ε is sufficiently small. Let $(\tilde{\Delta}, \tilde{\Delta}_{u_1}, \tilde{\Delta}_{u_2}, \tilde{\mu})$ generate $\overline{\varepsilon}_3^* > 0$, which is given below, such that $\tilde{\mu} > \underline{\alpha}_W^{-1} \circ \overline{\alpha}_W \circ \tilde{\alpha}_W^{-1}\left(\varepsilon\frac{C}{\zeta_3}\right)$ holds for all $\varepsilon \in (0, \overline{\varepsilon}_3^*)$. Hence, define

$$\overline{\varepsilon}^* := \min\{\overline{\varepsilon}_1^*, \overline{\varepsilon}_2^*, \overline{\varepsilon}_3^*\}, \tag{B.27}$$

with

$$\overline{\varepsilon}_1^* := \frac{\frac{2}{3}\zeta_1\zeta_3}{b_1(b_2+b_3) + \zeta_1(a_2+a_3) + \frac{2}{3}\zeta_3\left(a_1 + \frac{1}{4}\right)}, \tag{B.28a}$$

$$\overline{\varepsilon}_2^* := \frac{\zeta_3}{2(a_2+a_3+1)}, \tag{B.28b}$$

$$\overline{\varepsilon}_3^* := \frac{\zeta_3[\tilde{\alpha}_W^{-1} \circ \overline{\alpha}_W^{-1} \circ \underline{\alpha}_W(\tilde{\mu})]}{4[(b_2+b_3)^2\alpha_{V_1}^2(\tilde{\Delta}_x) + (\gamma_2(\tilde{\Delta}_{u_1}) + \gamma_3(\tilde{\Delta}_{u_1}))^2 + \gamma_4^2(\tilde{\Delta}_{u_2})]}, \tag{B.28c}$$

where all the above constants and functions come from Assumption 4.3 - 4.5. Since we have applied Lemma 4.1, we have that $\overline{\varepsilon}_1^*$ comes from $\tilde{\varepsilon}^*$ in (B.5). And $\overline{\varepsilon}_2^*$, $\overline{\varepsilon}_3^*$ are constructed as described above. We now consider the Lyapunov function $W(t, x, \xi)$ in Assumption 4.4 and take its derivative along the solutions of (4.6). This leads to

$$\dot{W}\big|_{(4.6)} = \frac{\partial W}{\partial t} + \frac{\partial W}{\partial x} f_s + \frac{1}{\varepsilon}\frac{\partial W}{\partial \xi} f_f - \frac{\partial W}{\partial \xi}\left(\frac{\partial H}{\partial t} + \frac{\partial H}{\partial x} f_s + \frac{\partial H}{\partial u}\dot{u}\right). \tag{B.29}$$

By adding and subtracting terms, and using inequalities in Assumptions 4.4 and 4.5 we obtain

$$\begin{aligned}\dot{W}\big|_{(4.6)} \leq& -\frac{1}{\varepsilon}\zeta_3\alpha_W^2(|\xi|) + (a_2+a_3)\alpha_W^2(|\xi|) + [(b_2+b_3)\alpha_{V_1}(|x|) + \gamma_2(|u|) \\ &+ \gamma_3(|u|) + \gamma_4(|\dot{u}|)]\alpha_W(|\xi|).\end{aligned} \tag{B.30}$$

Applying completion of squares to (B.30) leads to

$$\begin{aligned}\dot{W}\big|_{(4.6)} \leq& -\frac{1}{\varepsilon}\zeta_3\alpha_W^2(|\xi|) + (a_2+a_3+1)\alpha_W^2(|\xi|) + (b_2+b_3)^2\alpha_{V_1}^2(|x|) + \gamma_2^2(|u|) \\ &+ \gamma_3^2(|u|) + \gamma_4^2(|\dot{u}|).\end{aligned} \tag{B.31}$$

Then, we have that

$$\dot{W}\big|_{(4.6)} \leq -\frac{1}{2\varepsilon}\zeta_3\alpha_W^2(|\xi|) + (b_2+b_3)^2\alpha_{V_1}^2(|x|) + \gamma_2^2(|u|) + \gamma_3^2(|u|) + \gamma_4^2(|\dot{u}|), \tag{B.32}$$

for all $\varepsilon \in (0, \overline{\varepsilon}_2^*)$ with $\overline{\varepsilon}_2^*$ given by (B.28b). Note that (4.10) in Assumption 4.4 and (B.32) lead to

$$|\xi| \geq \tilde{\alpha}_W^{-1}\left(\varepsilon\frac{4}{\zeta_3}\left[\tilde{b}\alpha_{V_1}^2(|x|) + \gamma_2^2(|u|) + \gamma_3^2(|u|) + \gamma_4^2(|\dot{u}|)\right]\right) \implies \dot{W}\big|_{(4.6)} \leq -\frac{1}{\varepsilon}\hat{\alpha}_W(W), \tag{B.33}$$

where $\tilde{b} = (b_2+b_3)^2$ and $\hat{\alpha}_W(\cdot) = \frac{\zeta_3}{4}\alpha_W^2 \circ \overline{\alpha}_W^{-1}(\cdot)$. To conclude an ISS result, we have from [Theorem 4.18, 70] that the following condition must hold

$$\tilde{\alpha}_W^{-1}\left(\varepsilon\frac{4}{\zeta_3}[(b_2+b_3)^2\alpha_{V_1}^2(\tilde{\Delta}_x) + \gamma_2^2(\tilde{\Delta}_{u_1}) + \gamma_3^2(\tilde{\Delta}_{u_1}) + \gamma_4^2(\tilde{\Delta}_{u_2})]\right) \leq \tilde{\Delta}, \tag{B.34}$$

where $\tilde{\alpha}_W(\cdot) := \alpha_W^2(\cdot)$. Note that we consider $x(t)$ as an input to the $\xi - subsystem$. It is observed from (B.33) that the fast dynamics are ISS with respect to x, u and $\dot{u}$. It follows from Lemma 4.1 that $x(t)$ is an input bounded signal to the $\xi-subsystem$. Note that we have that $x, u, \dot{u} \in \mathcal{L}_\infty$. Hence, by applying results in [114] and [116] to (B.33), it follows that

$$|\xi(t)| \leq \beta_\xi\left(|\xi_0|, \frac{t-t_0}{\varepsilon}\right) + \underline{\alpha}_W^{-1} \circ \overline{\alpha}_W \circ \tilde{\alpha}_W^{-1}\left(\varepsilon\frac{4}{\zeta_3}\left[(b_2+b_3)^2\alpha_{V_1}^2(|x[t_0,t]|) \right.\right.$$
$$\left.\left. +\gamma_2^2(|u[t_0,t]|) + \gamma_3^2(|u[t_0,t]|) + \gamma_4^2(|\dot{u}[t_0,t]|)\right]\right). \tag{B.35}$$

Note that $4\left[(b_2+b_3)^2\alpha_{V_1}^2(|x[t_0,t]|) + \gamma_2^2(|u[t_0,t]|) + \gamma_3^2(|u[t_0,t]|) + \gamma_4^2(|\dot{u}[t_0,t]|)\right] \leq C$ with C given by (B.25). This and (B.35) lead to

$$|\xi(t)| \leq \beta_\xi\left(|\xi_0|, \frac{t-t_0}{\varepsilon}\right) + \underline{\alpha}_W^{-1} \circ \overline{\alpha}_W \circ \tilde{\alpha}_W^{-1}\left(\varepsilon\frac{C}{\zeta_3}\right). \tag{B.36}$$

Since the second term on the right-hand side of (B.36) is equal to $\tilde{C}$ and $\tilde{\mu} > \tilde{C}$ holds for all $\varepsilon \in (0, \overline{\varepsilon}_3^*)$ with $\overline{\varepsilon}_3^*$ given by (B.28c), we conclude that (4.19) holds for all $\varepsilon \in (0, \overline{\varepsilon}^*)$ and for all $|(x_0, \xi_0)| \leq \tilde{\Delta}$, $||u||_\infty \leq \tilde{\Delta}_{u_1}$, $||\dot{u}||_\infty \leq \tilde{\Delta}_{u_2}$ and $t \geq t_0 \geq 0$.

Step 2) We now define

$$T^* := \varepsilon\frac{\upsilon - \nu}{K}, \tag{B.37}$$

where $\upsilon := \overline{\alpha}_W \circ \tilde{\alpha}_W^{-1}\left(\varepsilon\frac{4}{\zeta_3}[(b_2+b_3)^2\alpha_{V_1}^2(\tilde{\Delta}_x) + \gamma_2^2(\tilde{\Delta}_{u_1}) + \gamma_3^2(\tilde{\Delta}_{u_1}) + \gamma_4^2(\tilde{\Delta}_{u_2})]\right)$, $\nu := \overline{\alpha}_W(\tilde{\Delta})$ and $K = \min\{\hat{\alpha}_W(W)\}$ over the set $\{\tilde{\Delta} \leq |\xi| \leq \tilde{\Delta}_\xi\}$ with $\tilde{\Delta}_\xi = \underline{\alpha}_W^{-1} \circ \overline{\alpha}_W(\tilde{\Delta})$. Note that T^* is of $O(\varepsilon)$. To prove that the second part of the corollary holds, we consider the inequality on the right-hand side of (B.33) which implies that

$$\dot{W}\big|_{(6.48)} \leq -\frac{1}{\varepsilon}K < 0. \tag{B.38}$$

Integrating (B.38) leads to

$$W(t, x(t), \xi(t)) \leq W(t_0, x(t_0), \xi(t_0)) - \frac{1}{\varepsilon}K(t-t_0). \tag{B.39}$$

Note that $\overline{\alpha}_W(|\xi_0|) \leq \nu$. So, it follows from (4.10) that

$$W(t, x(t), \xi(t)) \leq \nu - \frac{1}{\varepsilon}K(t-t_0). \tag{B.40}$$

So, we have from (B.34) that $\upsilon \leq \nu$. It is observed that (B.40) implies there is a time $t > 0$ such that $\upsilon = \nu - \frac{1}{\varepsilon}K(t - t_0)$. From this equality, we conclude that such a time is $t = T^* + t_0$ where T^* is given by (B.37). Moreover, we have that

$$\upsilon \geq \nu - \frac{1}{\varepsilon}K(t - t_0), \tag{B.41}$$

for all $t \in [t_0 + T^*, \infty)$. Then, it follows from (B.40) and (B.41) that (4.20) holds for all $\varepsilon \in (0, \overline{\varepsilon}^*)$, and for all $|(x_0, \xi_0)| \leq \tilde{\Delta}$, $||u||_\infty \leq \tilde{\Delta}_{u_1}$, $||\dot{u}||_\infty \leq \tilde{\Delta}_{u_2}$ and $t \geq T^* + t_0$.

Step 3) We now prove the practical $\mathcal{L}_2$ stability result represented by (4.21). Define the $\mathcal{K}_\infty$ function

$$\alpha_{W_c}(s) = \frac{2}{\zeta_3}\overline{\alpha}_W(s). \tag{B.42}$$

Let $|(x_0, \xi_0)| \leq \tilde{\Delta}$, $||u||_{\mathcal{L}_2} \leq \tilde{\Delta}_{u_1}$, and $||\dot{u}||_{\mathcal{L}_2} \leq \tilde{\Delta}_{u_2}$. Define

$$\tilde{\mu}_{\mathcal{L}_2} := \frac{C}{2\zeta_3}, \tag{B.43}$$

where C is given by (B.25) and ζ_3 come from Assumption 4.4. Define

$$\overline{\varepsilon}^*_{\mathcal{L}_2} := \min\{\overline{\varepsilon}^*_1, \overline{\varepsilon}^*_2\}, \tag{B.44}$$

where $\overline{\varepsilon}^*_1$ and $\overline{\varepsilon}^*_2$ are given by (B.28a) and (B.28b), respectively. We now bound (B.32) by considering $|x| \leq \tilde{\Delta}_x$, $|u| \leq \tilde{\Delta}_{u_1}$ and $|\dot{u}| \leq \tilde{\Delta}_{u_2}$ as follows

$$\dot{W}\big|_{(4.6)} \leq -\frac{1}{2\varepsilon}\zeta_3\alpha_W^2(|\xi|) + (b_2 + b_3)^2\alpha_{V_1}^2(\tilde{\Delta}_x) + \gamma_2^2(\tilde{\Delta}_{u_1}) + \gamma_3^2(\tilde{\Delta}_{u_1}) + \gamma_4^2(\tilde{\Delta}_{u_2}). \tag{B.45}$$

We integrate both sides of (B.45), which leads to

$$\begin{aligned} W(t, x(t), \xi(t)) - W(t_0, x(t_0), \xi(t_0)) \leq -\frac{1}{2\varepsilon}\zeta_3\int_{t_0}^{t} \alpha_W^2(|\xi(\tau)|)d\tau + \Big[(b_2 + b_3)^2\alpha_{V_1}^2(\tilde{\Delta}_x) \\ +\gamma_2^2(\tilde{\Delta}_{u_1}) + \gamma_3^2(\tilde{\Delta}_{u_1}) + \gamma_4^2(\tilde{\Delta}_{u_2})\Big]\int_{t_0}^{t} d\tau. \end{aligned} \tag{B.46}$$

We use the fact that $W(t, x(t), \xi(t)) \geq 0$ to obtain

$$\begin{aligned} \frac{1}{2\varepsilon}\zeta_3\int_{t_0}^{t} \alpha_W^2(|\xi(\tau)|)d\tau \leq W(t_0, x(t_0), \xi(t_0)) + \Big[(b_2 + b_3)^2\alpha_{V_1}^2(\tilde{\Delta}_x) + \gamma_2^2(\tilde{\Delta}_{u_1}) \\ +\gamma_3^2(\tilde{\Delta}_{u_1}) + \gamma_4^2(\tilde{\Delta}_{u_2})\Big]\int_{t_0}^{t} d\tau. \end{aligned} \tag{B.47}$$

Note that (4.10) implies $W(t_0, x(t_0), \xi(t_0)) \leq \overline{\alpha}_W(|\xi_0|)$. So, it follows that

$$\int_{t_0}^{t} \alpha_W^2(|\xi(\tau)|)d\tau \leq \varepsilon\frac{2}{\zeta_3}\overline{\alpha}_W(|\xi_0|) + \varepsilon\frac{2}{\zeta_3}\Big[(b_2+b_3)^2\alpha_{V_1}^2(\tilde{\Delta}_x) + \gamma_2^2(\tilde{\Delta}_{u_1}) + \gamma_3^2(\tilde{\Delta}_{u_1}) + \gamma_4^2(\tilde{\Delta}_{u_2})\Big]\int_{t_0}^{t} d\tau. \tag{B.48}$$

By using $\alpha_{W_c}(\cdot)$ and $\tilde{\mu}_{\mathcal{L}_2}$ defined as in (B.42) and (B.43) respectively, it follows from (B.48) that (4.21) holds for all $\varepsilon \in (0, \overline{\varepsilon}^*_{\mathcal{L}_2})$, and for all $|(x_0, \xi_0)| \leq \tilde{\Delta}$, $\|u\|_{\mathcal{L}_2} \leq \tilde{\Delta}_{u_1}$, $\|\dot{u}\|_{\mathcal{L}_2} \leq \tilde{\Delta}_{u_2}$ and $t \geq t_0 \geq 0$. This completes the proof. ■

B.3 Proof of Corollary 4.2

We split the proof into two steps. In the first step, we prove that the ultimate boundedness condition in (4.32) holds. Then, in the second step we show that the practical $\mathcal{L}_2$ stability property in (4.33) is satisfied.

Step 1) Define the class-$\mathcal{K}_\infty$ function

$$\alpha_{c_1}(s) := \alpha_{o_1}(s), \tag{B.49}$$

where $\alpha_{o_1}(\cdot)$ comes from Assumption 4.8. Let $\overline{\Delta} > 0$, $\overline{\Delta}_{u_1} > 0$,$\overline{\Delta}_{u_2} > 0$ and $\overline{\Delta}_w > 0$ be given such that $|(x_0, \xi_0, \chi_0)| \leq \overline{\Delta}$, $\|u\|_\infty \leq \overline{\Delta}_{u_1}$, $\|\dot{u}\|_\infty \leq \overline{\Delta}_{u_2}$ and $\|w\|_\infty \leq \overline{\Delta}_w$. By using Lemma 4.1, we generate $\tilde{\varepsilon}^* > 0$ such that (4.17) holds. Introduce $\overline{\Delta}_{(x,\xi)} := \beta_{L_1}\left(\overline{\Delta}, 0\right) + \gamma_{L_1}(\overline{\Delta}_{u_1}) + \tilde{\gamma}_{L_1}(\overline{\Delta}_{u_1}) + \hat{\gamma}_{L_1}(\overline{\Delta}_{u_2}) + \mu_{L_1}$, where $\beta_{L_1}(\cdot,\cdot)$, $\gamma_{L_1}(\cdot)$, $\tilde{\gamma}_\varepsilon(\cdot)$, $\hat{\gamma}_\varepsilon(\cdot)$ and μ_{L_1} come from Lemma 4.1. Then, (4.17) implies that $|(x(t), \xi(t))| \leq \Delta_{(x,\xi)}$ for all $|(x_0, \xi_0)| \leq \overline{\Delta}$, $u \in \overline{B}_{u_1}$, $\dot{u} \in \overline{B}_{u_2}$, and $t \geq t_0 \geq 0$ where $\overline{B}_{u_1} = \{u \in \mathbb{R}^r \mid |u| \leq \overline{\Delta}_{u_1}\}$ and $\overline{B}_{u_2} = \{\dot{u} \in \mathbb{R}^r \mid |\dot{u}| \leq \overline{\Delta}_{u_2}\}$. Define $(\hat{\Delta}, \hat{\Delta}_{u_1}, \hat{\Delta}_w)$ as $\hat{\Delta} := \overline{\Delta}_{(x,\xi)}$, $\hat{\Delta}_{u_1} := \overline{\Delta}_{u_1}$, $\hat{\Delta}_w := \overline{\Delta}_w$. By virtue of Assumption 4.9, let $(\hat{\Delta}, \hat{\Delta}_{u_1}, \hat{\Delta}_w)$ generate $\varepsilon_y > 0$ such that (4.31) holds. Introduce $\overline{\Delta}_h := \alpha_y(\hat{\Delta}) + \gamma_y(\hat{\Delta}_{u_1}) + \gamma_y(\hat{\Delta}_w)$ and note that (4.31) implies that $|h(t, x, \xi + H(t, x, u), u, \varepsilon)| \leq \overline{\Delta}_h$ for all $\varepsilon \in (0, \varepsilon_y)$ and for all $|(x, \xi)| \leq \hat{\Delta}$, $|u| \leq \hat{\Delta}_{u_1}$, $|w| \leq \hat{\Delta}_w$ and $t \geq 0$. Define

$$\hat{\varepsilon}^* := \min\left\{\frac{\frac{2}{3}\zeta_1\zeta_3}{b_1(b_2+b_3) + \zeta_1(a_2+a_3) + \frac{2}{3}\zeta_3\left(a_1 + \frac{1}{4}\right)}, \varepsilon_y\right\}, \tag{B.50}$$

where all of the above constants come from Assumptions 4.3 - 4.5. Finally, define

$$\Upsilon := \alpha_{o_2}(\overline{\Delta}_h) + \alpha_{o_3}(\overline{\Delta}_{u_1}), \tag{B.51}$$

where functions $\alpha_{o_2}(\cdot), \alpha_{o_3}(\cdot) \in \mathcal{K}_\infty$ come from Assumption 4.8. Since we have applied Lemma 4.1, we have that the first term in (B.50) corresponds to $\tilde{\varepsilon}^*$ in (B.5). Note that the second term in (B.50) comes from Assumption 4.9. It is observed that the observer dynamics is in cascade with the original system (4.6) through the output of the system. Then, the output is seen as a signal input to the observer. Since the map of the output is bounded for any $\varepsilon \in (0, \varepsilon_y)$, it follows that y is a bounded signal input to the observer dynamics, i.e., $\|y(t)\|_\infty \leq \overline{\Delta}_h$. Hence, we have from Assumption 4.8 that Υ in (B.51) is well defined. Moreover, we conclude from (4.27) that (4.32) holds for any $\varepsilon \in (0, \hat{\varepsilon}^*)$ and for all $|(x_0, \xi_0, \chi_0)| \leq \overline{\Delta}$, $\|u\|_\infty \leq \overline{\Delta}_{u_1}$, $\|\dot{u}\|_\infty \leq \overline{\Delta}_{u_2}$, $\|w\|_\infty \leq \overline{\Delta}_w$ and $t \geq t_0 \geq 0$.

Step 2) We now prove that (4.33) holds. Define

$$\alpha_{c_2}(s) := \alpha_{o_4}(s), \tag{B.52}$$

$$\alpha_{c_3}(s) := \alpha_{o_5}(s), \tag{B.53}$$

where α_{o_i}, for $i \in \{4, 5\}$, come from Assumption 4.8. Let $\|u\|_{\mathcal{L}_2} \leq \overline{\Delta}_{u_1}$, $\|\dot{u}\|_{\mathcal{L}_2} \leq \overline{\Delta}_{u_2}$ and $\|w\|_{\mathcal{L}_2} \leq \overline{\Delta}_w$. It follows that Lemma 2.1 holds and $\|y\|_{\mathcal{L}_2} \leq \overline{\Delta}_h$. Define

$$\Upsilon_{\mathcal{L}_2} := \alpha_{o_6}(\overline{\Delta}_h) + \alpha_{o_7}(\overline{\Delta}_{u_1}), \tag{B.54}$$

where α_{o_i} $(i = 6, 7)$ come from Assumption 4.8. Then, it follows from (4.28) and the linearity of the integral function that

$$\int_{t_0}^{t} \alpha_{c_2}(|\chi(\tau)|)d\tau \leq \alpha_{c_3}(|\chi_0|) + \Upsilon_{\mathcal{L}_2} \int_{t_0}^{t} d\tau. \tag{B.55}$$

Therefore, we conclude from (B.55) that (4.33) holds for all $\varepsilon \in (0, \hat{\varepsilon}^*)$, $|(x_0, \xi_0, \chi_0)| \leq \overline{\Delta}$, $\|u\|_{\mathcal{L}_2} \leq \overline{\Delta}_{u_1}$, $\|\dot{u}\|_{\mathcal{L}_2} \leq \overline{\Delta}_{u_2}$, $\|w\|_{\mathcal{L}_2} \leq \Delta_w$ and $t \geq t_0 \geq 0$. This completes the proof. ■

B.4 Proof of Lemma 4.2

We split the proof in two steps. In the first step, we prove that (4.38) holds under Assumptions 4.1 - 4.10. We then show that the error dynamics satisfy (4.39).

Step 1) Let Assumptions 4.1 - 4.10 hold. Let $\lambda_e(\cdot,\cdot) \in \mathcal{KL}$ be defined as the solution of the following scalar differential equation,

$$\dot{y_e} = -\hat{\alpha}_{V_3}(y_e), \;\; y_e(t_0) = y_{e_0}, \tag{B.56}$$

with $\hat{\alpha}_{V_3}(\cdot) = \frac{1}{4}\zeta_2\alpha^2_{V_3} \circ \overline{\alpha}^{-1}_{V_3}(\cdot)$ where ζ_2, $\alpha_{V_3}(\cdot)$ and $\overline{\alpha}_{V_3}(\cdot)$ come from Assumption 6.7. Then, $y(t) = \lambda_e(y_{e_0}, t-t_0)$. The existence of $\lambda_e(\cdot,\cdot)$ follows from [Lemma 4.4, 70]. Define the class-$\mathcal{KL}$ function

$$\beta_e(r,s) := \underline{\alpha}^{-1}_{V_3}(\lambda_e(\overline{\alpha}_{V_3}(r), s)), \tag{B.57}$$

where the functions $\underline{\alpha}_{V_3}(\cdot)$ and $\overline{\alpha}_{V_3}(\cdot)$ come from Assumption 4.7. Define

$$\gamma_\xi(s) := \underline{\alpha}^{-1}_{V_3} \circ \overline{\alpha}_{V_3} \circ \tilde{\alpha}^{-1}_{V_3}\left(\frac{8(b_4+b_5+b_6+Lb_7)^2}{\zeta_2^2}\alpha^2_W(s)\right), \tag{B.58a}$$

$$\gamma_{x,\varepsilon}(s) := \underline{\alpha}^{-1}_{V_3} \circ \overline{\alpha}_{V_3} \circ \tilde{\alpha}^{-1}_{V_3}\left(\varepsilon\frac{8(a_4+a_5+a_6+La_7)}{\zeta_2}\alpha^2_{V_1}(s)\right), \tag{B.58b}$$

$$\gamma_{u,\varepsilon}(s) := \underline{\alpha}^{-1}_{V_3} \circ \overline{\alpha}_{V_3} \circ \tilde{\alpha}^{-1}_{V_3}\left(\varepsilon\frac{8}{\zeta_2}\left[\gamma_5^2(s)+\gamma_6^2(s)\right]\right), \tag{B.58c}$$

$$\gamma_w(s) := \underline{\alpha}^{-1}_{V_3} \circ \overline{\alpha}_{V_3} \circ \tilde{\alpha}^{-1}_{V_3}\left(\frac{8}{\zeta_2}\gamma_{V_3}(s)\right), \tag{B.58d}$$

where $\tilde{\alpha}_{V_3}(\cdot) = \alpha^2_{V_3}(\cdot)$, and all constants and functions come from Assumptions 4.3 - 4.6, 4.9 and 4.10, and $L > 0$ is such that $|\partial h_o/\partial\chi| \leq L$ for all $\chi \in B_1$ with $B_1 = \{\chi \in \mathbb{R}^q \mid |\chi| \leq \Delta_1\}$ where $\Delta_1 > 0$. Let $\Delta_L > 0$, $\Delta_{L_{u_1}} > 0$, $\Delta_{L_{u_2}} > 0$, and $\Delta_{L_w} > 0$ be given such that $|(x_0,\xi_0,\chi_0,e_0)| \leq \Delta_L$, $\|u\|_\infty \leq \Delta_{L_{u_1}}$, $\|\dot{u}\|_\infty \leq \Delta_{L_{u_2}}$, and $\|w\|_\infty \leq \Delta_{L_w}$. By using Lemma 4.1, we generate $\tilde{\varepsilon}^* > 0$ such that (4.17) holds for all $\varepsilon \in (0,\tilde{\varepsilon}^*)$. Define $(\overline{\Delta},\overline{\Delta}_{u_1},\overline{\Delta}_{u_2},\overline{\Delta}_w)$ as $\overline{\Delta} := \Delta_L$, $\overline{\Delta}_{u_1} := \Delta_{L_{u_1}}$, $\overline{\Delta}_{u_2} := \Delta_{L_{u_2}}$, $\overline{\Delta}_w := \Delta_{L_w}$. By using Corollary 4.2, let $(\overline{\Delta},\overline{\Delta}_{u_1},\overline{\Delta}_{u_2},\overline{\Delta}_w)$ generate $\hat{\varepsilon}^* > 0$ and $\Upsilon > 0$ such that (4.32) holds for all $\varepsilon \in (0,\hat{\varepsilon}^*)$.

We now introduce $\Delta_x := \beta_{L_1}(\Delta_L, 0)+\gamma_{L_1}(\Delta_{L_{u_1}})+\tilde{\gamma}_{L_1}(\Delta_{L_{u_1}})+\hat{\gamma}_{L_1}(\Delta_{L_{u_2}})+\mu_{L_1}$ and $\Delta_\chi := \alpha_{c_1}(\overline{\Delta}) + \Upsilon$, where $\beta_{L_1}(\cdot,\cdot)$, $\gamma_{L_1}(\cdot)$, $\tilde{\gamma}_{L_1}(\cdot)$, $\hat{\gamma}_{L_1}(\cdot)$ and μ_{L_1} come from (4.17) in Lemma 4.1, and $\alpha_{c_1}(\cdot)$ and Υ come from (4.32) in Corollary 4.2. Then, we have that $|x(t)| \leq \Delta_x$ for all $|(x_0,\xi_0)| \leq \Delta_L$, $u \in B_{u_1}$, $\dot{u} \in B_{u_2}$, and $t \geq t_0 \geq 0$ where $B_{u_1} = \{u \in \mathbb{R}^r \mid |u| \leq \Delta_{L_{u_1}}\}$ and $B_{u_2} = \{\dot{u} \in \mathbb{R}^r \mid |\dot{u}| \leq \Delta_{L_{u_2}}\}$. From Lemma 4.1, we have that $|(x(t),\xi(t))| \leq \Delta_x$ for all $|(x_0,\xi_0)| \leq \Delta_L$, $u \in B_{u_1}$, $\dot{u} \in B_{u_2}$, and $t \geq t_0 \geq 0$. Moreover, from the choice of $(\overline{\Delta},\overline{\Delta}_{u_1},\overline{\Delta}_{u_2},\overline{\Delta}_w)$, we conclude that $|\chi(t)| \leq \Delta_\chi$ for all $|(x_0,\xi_0,\chi_0)| \leq \overline{\Delta}$, $u \in B_{u_1}$, $\dot{u} \in B_{u_2}$, and $t \geq t_0 \geq 0$. Let $\varepsilon^*_{L_3} > 0$ be such that $\varepsilon(a_4+a_5+a_6+La_7+2) - \zeta_2 < 0$ for all $\varepsilon \in (0,\varepsilon^*_{L_3})$, $\varepsilon^*_{L_3}$ is given below. Hence, define

$$\varepsilon^*_L := \min\{\varepsilon^*_{L_1}, \varepsilon^*_{L_2}, \varepsilon^*_{L_3}\}, \tag{B.59}$$

with

$$\varepsilon_{L_1}^* := \frac{\frac{2}{3}\zeta_1\zeta_3}{b_1(b_2+b_3)+\zeta_1(a_2+a_3)+\frac{2}{3}\zeta_3\left(a_1+\frac{1}{4}\right)}, \tag{B.60a}$$

$$\varepsilon_{L_2}^* := \varepsilon_y, \tag{B.60b}$$

$$\varepsilon_{L_3}^* := \frac{\zeta_2}{a_4+a_5+a_6+La_7+2}, \tag{B.60c}$$

where all of the above constants come from Assumptions 4.3 - 4.6, 4.9 and 4.10.

Note that (B.60a) comes from $\tilde{\varepsilon}^*$ in Lemma 4.1. Moreover, from the choice of $(\overline{\Delta}, \overline{\Delta}_{u_1}, \overline{\Delta}_{u_2}, \overline{\Delta}_w)$ and Corollary 4.2, we have that (B.60b) comes from $\hat{\varepsilon}^*$. We have introduced above the condition from which $\varepsilon_{L_3}^*$ has been constructed. To prove that (4.38) holds, we now consider the Lyapunov function $V_3(t,e,x,\chi)$ in Assumption 4.7 and take its derivative along the solutions of (4.30), which is given by

$$\begin{aligned}\dot{V}_3\big|_{(4.30)} &= \frac{\partial V_3}{\partial t} + \frac{\partial V_3}{\partial e} f_e(t,x,\chi,e,\xi+H(t,x,u),y,u,\dot{u},\varepsilon)\\ &\quad + \frac{\partial V_3}{\partial x} f_s(t,x,\xi+H(t,x,u),u,\varepsilon) + \frac{\partial V_3}{\partial \chi} f_o(t,\chi,y,u).\end{aligned} \tag{B.61}$$

By adding and subtracting terms and using the definition of the error dynamics, we can rewrite (B.61) as follows

$$\begin{aligned}\dot{V}_3\big|_{(4.30)} &= \frac{\partial V_3}{\partial t} + \frac{\partial V_3}{\partial e} f_e(t,x,\chi,e,\xi+H(t,x,u),y_s,u,\dot{u},0) + \frac{\partial V_3}{\partial x} f_s(t,x,H(t,x,u),u,0)\\ &\quad + \frac{\partial V_3}{\partial \chi} f_o(t,\chi,y_s,u) + \frac{\partial V_3}{\partial x}\Big[f_s(t,x,\xi+H(t,x,u),u,\varepsilon) - f_s(t,x,H(t,x,u),u,0)\Big]\\ &\quad + \frac{\partial V_3}{\partial \chi}\Big[f_o(t,\chi,y,u) - f_o(t,\chi,y_s,u)\Big] + \frac{\partial V_3}{\partial e}\left[\frac{\partial h_o}{\partial \chi}\Big[f_o(t,\chi,y,u) - f_o(t,\chi,y_s,u)\Big]\right]\\ &\quad + \frac{\partial V_3}{\partial e}\left[f_s(t,x,H(t,x,u),u,0) - f_s(t,x,\xi+H(t,x,u),u,\varepsilon)\right].\end{aligned} \tag{B.62}$$

As showed above, it follows from Corollary 4.2 that $|\chi| \le \Delta_\chi$ for all $\varepsilon \in (0,\hat{\varepsilon}^*)$ where $\hat{\varepsilon}^* \le \varepsilon_L^*$. Let define $\Delta_1 := \Delta_\chi$ and $\Delta_2 := \Delta_{L_{u_1}}$, so we have from Remark 4.1 that for the given Δ_1 and Δ_2 there is $L > 0$ such that $|\partial h_o/\partial\chi| \le L$ for all $\chi \in B_1$ with $B_1 = \{\chi \in \mathbb{R}^q \mid |\chi| \le \Delta_1\}$ where $\Delta_1 := \Delta_\chi$. By using the norm and applying inequalities in Assumptions 4.7 and 4.10, we have

$$\begin{aligned}\dot{V}_3\big|_{(4.30)} &\le -\zeta_2\alpha_{V_3}^2(|e|) + \varepsilon(a_4+a_5+a_6+La_7)\alpha_{V_1}(|x|)\alpha_{V_3}(|e|) + \varepsilon\gamma_5(|u|)\alpha_{V_3}(|e|)\\ &\quad + (b_4+b_5+b_6+Lb_7)\alpha_{V_3}(|e|)\alpha_W(|\xi|) + \varepsilon\gamma_6(|u|)\alpha_{V_3}(|e|) + \gamma_{V_3}(|w|).\end{aligned} \tag{B.63}$$

Applying completion of squares to (B.63) leads to

$$\dot{V}_3\big|_{(4.30)} \leq -\frac{3}{4}\zeta_2\alpha^2_{V_3}(|e|) + \varepsilon\frac{1}{4}(a_4 + a_5 + a_6 + La_7 + 2)\alpha^2_{V_3}(|e|) + \varepsilon\gamma^2_5(|u|) + \varepsilon\gamma^2_6(|u|)$$
$$+ \varepsilon(a_4 + a_5 + a_6 + La_7)\alpha^2_{V_1}(|x|) + \frac{\overline{k}_1}{\zeta_2}\alpha^2_W(|\xi|) + \gamma_{V_3}(|w|), \tag{B.64}$$

with $\overline{k}_1 = (b_4 + b_5 + b_6 + Lb_7)^2$. It follows from (B.64) that

$$\dot{V}_3\big|_{(4.30)} \leq -\frac{1}{2}\zeta_2\alpha^2_{V_3}(|e|) + \varepsilon(a_4 + a_5 + a_6 + La_7)\alpha^2_{V_1}(|x|) + \varepsilon\gamma^2_5(|u|) + \varepsilon\gamma^2_6(|u|)$$
$$+ \frac{\overline{k}_1}{\zeta_2}\alpha^2_W(|\xi|) + \gamma_{V_3}(|w|), \tag{B.65}$$

for all $\varepsilon \in (0, \varepsilon^*_{L_3})$ with $\varepsilon^*_{L_3}$ given by (B.60c). It is observed that $\varepsilon^*_{L_3} \leq \varepsilon^*_L$. Then, it follows from (4.24) and (B.65) that

$$|e| \geq \tilde{\alpha}^{-1}_{V_3}\left(\frac{4}{\zeta_2}\Big[\varepsilon(a_4 + a_5 + a_6 + La_7)\alpha^2_{V_1}(|x|) + \varepsilon\gamma^2_5(|u|) + \varepsilon\gamma^2_6(|u|)\right.$$
$$\left. + \frac{\overline{k}_1}{\zeta_2}\alpha^2_W(|\xi|) + \gamma_{V_3}(|w|)\Big]\right) \implies \dot{V}_3\big|_{(4.30)} \leq -\hat{\alpha}_{V_3}(V_3), \tag{B.66}$$

where $\tilde{\alpha}_{V_3}(\cdot) = \alpha^2_{V_3}(\cdot)$ and $\hat{\alpha}_{V_3}(\cdot) = \frac{1}{4}\zeta_2\alpha^2_{V_3} \circ \overline{\alpha}^{-1}_{V_3}(\cdot)$. We can conclude an ISS result from (B.66) if the following condition holds

$$\tilde{\alpha}^{-1}_{V_3}\left(\frac{4}{\zeta_2}\Big[\varepsilon(a_4 + a_5 + a_6 + La_7)\alpha^2_{V_1}(\Delta_x) + \varepsilon\gamma^2_5(\Delta_{L_{u_1}}) + \varepsilon\gamma^2_6(\Delta_{L_{u_1}})\right.$$
$$\left. + \frac{\overline{k}_1}{\zeta_2}\alpha^2_W(\Delta_x) + \gamma_{V_3}(\Delta_{L_w})\Big]\right) \leq \Delta_L, \tag{B.67}$$

for any $(x, \xi, \chi, e) \in B_\rho$, where $B_\rho := \{(x, \xi, \chi, e) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^n \mid |(x, \xi, \chi, e)| \leq \underline{\alpha}^{-1} \circ \overline{\alpha}(\Delta_L)\}$. If (B.67) does not hold, the solutions would not belong to the invariant set that agrees with (4.24) and the dissipation inequality (B.65), see [Theorem 4.18, 70]. We now exploit the cascade properties of the error dynamics, which are in cascade with the x, ξ and χ. Since Lemma 4.1 and Corollary 4.2 hold, $|x(t)| \leq |(x(t), \xi(t))|$ and $|\xi(t)| \leq |(x(t), \xi(t))|$, it follows that x, ξ and χ are essentially bounded inputs to the error dynamics. Then, (B.66) implies that the error dynamics are ISS with respect to x, ξ, u, and w. By applying results in [114] and [116], we obtain

$$|e(t)| \leq \beta_e(|e_0|, t - t_0) + \underline{\alpha}^{-1}_{V_3} \circ \overline{\alpha}_{V_3} \circ \tilde{\alpha}^{-1}_{V_3}\left(\frac{4}{\zeta_2}\Big[\varepsilon(a_4 + a_5 + a_6 + La_7)\alpha^2_{V_1}(|x[t_0, t]|)\right.$$

$$+\varepsilon\gamma_5^2(|u[t_0,t]|) + \varepsilon\gamma_6^2(|u[t_0,t]|) + \frac{\overline{k}_1}{\zeta_2}\alpha_W^2(|\xi[t_0,t]|) + \gamma_{V_3}(|w[t_0,t]|)\Big]\Big), \tag{B.68}$$

where $\beta_e(\cdot,\cdot) \in \mathcal{KL}$ is given by (B.57). By applying the weak triangle inequality to the second term on the right-hand side of (B.68), we conclude that (4.38) holds for all $\varepsilon \in (0, \varepsilon_L^*)$ and for all $|(x_0, \xi_0, \chi_0, e_0)| \le \Delta_L$, $\|u\|_\infty \le \Delta_{L_{u_1}}$, $\|\dot{u}\|_\infty \le \Delta_{L_{u_2}}$, $\|w\|_\infty \le \Delta_{L_w}$ and $t \ge t_0 \ge 0$, where $\gamma_\xi(\cdot)$, $\gamma_{x,\varepsilon}(\cdot)$, $\gamma_{u,\varepsilon}(\cdot)$, and $\gamma_w(\cdot)$ are given by (B.58).

Step 2) We now prove that (4.39) holds. Define

$$k_1 := \frac{2}{\zeta_2}, \tag{B.69a}$$

$$k_2 := \frac{2}{\zeta_2}(a_4 + a_5 + a_6 + La_7), \tag{B.69b}$$

$$k_3 := \frac{2(b_4 + b_5 + b_6 + Lb_7)^2}{\zeta_2^2}, \tag{B.69c}$$

where all constants come from Assumptions 4.7 and 4.10, and L is defined as in Step 1) of this proof. Let $\|u\|_{\mathcal{L}_2} \le \Delta_{L_{u_1}}$, $\|\dot{u}\|_{\mathcal{L}_2} \le \Delta_{L_{u_2}}$, and $\|w\|_{\mathcal{L}_2} \le \Delta_{L_w}$. Consider the set $\Omega_1 = \{(x,\xi,\chi,e) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^n \mid |(x,\xi,\chi,e)| \le \overline{\alpha}(\Delta_L)\}$ which is a subset of $B_\rho = \{(x,\xi,\chi,e) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^n \mid |(x,\xi,\chi,e)| \le \underline{\alpha}^{-1} \circ \overline{\alpha}(\Delta_L)\}$.

Claim: If $(x_0, \xi_0, \chi_0, e_0) \in \Omega_1$ for some $t_0 \ge 0$, then $(x(t), \xi(t), \chi(t), e(t)) \in \Omega_1$ for all $t \ge t_0$. *Proof of claim*: We prove our claim by contradiction. Assume there exists $\nu > 0$ and some $t_1 > t_0$ such that

$$V_3(t_1, e(t_1), x(t_1), \chi(t_1)) \ge \overline{\alpha}(\Delta_L) + \nu.$$

Let t_1 be minimal value of t such that the above inequality holds (for a fixed ν). Hence, $V_3(t, e(t), x(t), \chi(t)) > \overline{\alpha}(\Delta_L)$ for some t close to t_1. Since $V_3(t, e(t), x(t), \chi(t)) > \overline{\alpha}(\Delta_L)$ and $|e| \ge \overline{\alpha}^{-1}(V_3)$, we have

$$|e| \ge \Delta_L.$$

Then, it follows from (B.67) that the inequality on the left-hand side of (B.66 holds for each t in the neighbourhood of t_1, and the continuous function $V_3(t, e(t), x(t), \chi(t))$ has negative derivative near t_1. Thus, $V_3(t, e(t), x(t), \chi(t)) > V_3(t_1, e(t_1), x(t_1), \chi(t_1))$ for some $t \in (t_0, t_1)$, contradicting minimality of t_1. Therefore, Ω_1 must indeed be invariant, as claimed. This completes the proof of the claim.

Since Ω_1 is an invariant set, we know that any trajectory starting within the set will remain in it, and subsequently, in B_ρ. This implies that the norm infinity of the estimation error will remain bounded for all $t \geq t_0$. Moreover, we know from Lemma 2.1 and Corollary 2.2 (see Step 1) of this proof) that the states of the system and the states of the observer have finite and bounded infinity norm for all $\varepsilon \in (0, \varepsilon_L^*)$, $|(x_0, \xi_0, \chi_0, e_0)| \leq \Delta_L$, $||u||_\infty \leq \Delta_{L_{u_1}}$, $||\dot{u}||_\infty \leq \Delta_{L_{u_2}}$, $||w||_\infty \leq \Delta_{L_w}$ and $t \geq t_0 \geq 0$. Therefore, we can now integrate (B.65) as follows

$$\begin{aligned} V_3(t, e(t), x(t), \xi(t)) - V_3(t_0, e(t_0), x(t_0), \xi(t_0)) &\leq -\frac{1}{2}\zeta_2 \int_{t_0}^{t} \alpha_{V_3}^2(|e(\tau)|)d\tau \\ &+ \varepsilon \int_{t_0}^{t} \gamma_5^2(|u(\tau)|)d\tau + \varepsilon(a_4 + a_5 + a_6 + La_7) \int_{t_0}^{t} \alpha_{V_1}^2(|x(\tau)|)d\tau \\ &+ \varepsilon \int_{t_0}^{t} \gamma_6^2(|u(\tau)|)d\tau + \frac{\overline{k}_1}{\zeta_2} \int_{t_0}^{t} \alpha_W^2(|\xi(\tau)|)d\tau + \int_{t_0}^{t} \gamma_{V_3}(|w(\tau)|)d\tau, \end{aligned} \tag{B.70}$$

where $x(t)$ and $\xi(t)$ are the solutions to (4.6). We use that $V_3(t, e(t), x(t), \xi(t)) \geq 0$ to obtain

$$\begin{aligned} \frac{1}{2}\zeta_2 \int_{t_0}^{t} \alpha_{V_3}^2(|e(\tau)|)d\tau &\leq V_3(t_0, e(t_0), x(t_0), \xi(t_0)) + \varepsilon(a_4 + a_5 + a_6 + La_7) \int_{t_0}^{t} \alpha_{V_1}^2(|x(\tau)|)d\tau \\ &+ \varepsilon \int_{t_0}^{t} \gamma_5^2(|u(\tau)|)d\tau + \varepsilon \int_{t_0}^{t} \gamma_6^2(|u(\tau)|)d\tau + \frac{\overline{k}_1}{\zeta_2} \int_{t_0}^{t} \alpha_W^2(|\xi(\tau)|)d\tau + \int_{t_0}^{t} \gamma_{V_3}(|w(\tau)|)d\tau, \end{aligned} \tag{B.71}$$

It follows from (4.24) that $V_3(t_0, e(t_0), x(t_0), \xi(t_0)) \leq \overline{\alpha}_{V_3}(|e_0|)$. Therefore, it follows from (B.71) that (4.39) holds for all $\varepsilon \in (0, \varepsilon_L^*)$, $|(x_0, \xi_0, \chi_0, e_0)| \leq \Delta_L$, for any input satisfying $||u||_\infty \leq \Delta_{L_{u_1}}$, $||\dot{u}||_\infty \leq \Delta_{L_{u_2}}$, $||u||_{\mathcal{L}_2} \leq \Delta_{L_{u_1}}$, $||\dot{u}||_{\mathcal{L}_2} \leq \Delta_{L_{u_2}}$, for any $||w||_\infty \leq \Delta_{L_w}$, $||w||_{\mathcal{L}_2} \leq \Delta_{L_w}$ and for all $t \geq t_0 \geq 0$, where k_i $(i = 1, 2, 3)$ are given by (B.69). This completes the proof. ■

B.5 Proof of Theorem 4.1

We split the proof in four steps. In the first step, we prove that (4.40) holds under Assumptions 4.1 - 4.10. Then, we show that (4.41) holds under the same assumptions. In the third step, we prove that the error dynamics satisfy (4.42). Finally, we show that the error dynamics also satisfy (4.43).

Step 1) Let Assumptions 4.1 - 4.10 hold. Define the class-$\mathcal{KL}$ function

$$\beta_{T_1}(r,s) := \beta_e\left(2\left[\beta_e\left(r,\frac{s}{2}\right)+\gamma_\xi\left(2\beta_\xi(r,0)\right)\right],\frac{s}{2}\right)+\gamma_\xi\left(2\beta_\xi\left(r,\frac{s}{2\varepsilon}\right)\right), \tag{B.72}$$

where $\beta_e(\cdot,\cdot)\in\mathcal{KL}$ and $\gamma_\xi(\cdot)\in\mathcal{K}_\infty$ come from Lemma 4.2, and $\beta_\xi(\cdot,\cdot)\in\mathcal{KL}$ is given in Corollary 4.1. Define the class-$\mathcal{K}_\infty$ function

$$\gamma_{T_1}(s) := \gamma_w(s)+\beta_e\left(2\gamma_w(s),0\right), \tag{B.73}$$

where $\gamma_w(\cdot)\in\mathcal{K}_\infty$ comes from Lemma 4.2. Let $\Delta>0$, $\Delta_{u_1}>0$, $\Delta_{u_2}>0$, $\Delta_w>0$ and $\mu>0$ be given such that $|(x_0,\xi_0,\chi_0,e_0)|\le\Delta$, $\|u\|_\infty\le\Delta_{u_1}$, $\|\dot{u}\|_\infty\le\Delta_{u_2}$, and $\|w\|_\infty\le\Delta_w$. Define

$$\mu_{T_1} := \beta_e(2\mu,0). \tag{B.74}$$

Let $(\overline{\Delta},\overline{\Delta}_{u_1},\overline{\Delta}_{u_2},\overline{\Delta}_w)$ be defined as $\overline{\Delta}:=\Delta$, $\overline{\Delta}_{u_1}:=\Delta_{u_1}$, $\overline{\Delta}_{u_2}:=\Delta_{u_2}$, $\overline{\Delta}_w:=\Delta_w$. Using Corollary 4.2, let $(\overline{\Delta},\overline{\Delta}_{u_1},\overline{\Delta}_{u_2},\overline{\Delta}_w)$ generate $\hat{\varepsilon}^*>0$ and $\Upsilon>0$ such that (4.32) holds for all $\varepsilon\in(0,\hat{\varepsilon}^*)$. We now introduce $\Delta_\chi:=\alpha_{c_1}(\overline{\Delta})+\Upsilon$, where $\alpha_{c_1}(\cdot)$ and $\Upsilon>0$ come from (4.32) in Corollary 4.2. From the choice of $(\overline{\Delta},\overline{\Delta}_{u_1},\overline{\Delta}_{u_2},\overline{\Delta}_w)$, we conclude that $|\chi(t)|\le\Delta_\chi$ for all $|(x_0,\xi_0,\chi_0,e_0)|\le\overline{\Delta}$, $u\in B_{u_1}$, $\dot{u}\in B_{u_2}$, and $t\ge t_0\ge 0$ where $B_{u_1}=\{u\in\mathbb{R}^r\mid |u|\le\Delta_{u_1}\}$ and $B_{u_2}=\{\dot{u}\in\mathbb{R}^r\mid|\dot{u}|\le\Delta_{u_2}\}$. Define $(\tilde{\Delta},\tilde{\Delta}_{u_1},\tilde{\Delta}_{u_2},\tilde{\mu})$ as $\tilde{\Delta}:=\Delta$, $\tilde{\Delta}_{u_1}:=\Delta_{u_1}$, $\tilde{\Delta}_{u_2}:=\Delta_{u_2}$, and

$$\tilde{\mu} := \alpha_W^{-1}\left(\sqrt{\frac{\zeta_2^2}{8[b_4+b_5+b_6+Lb_7]^2}\tilde{\alpha}_{V_3}\circ\overline{\alpha}_{V_3}^{-1}\circ\underline{\alpha}_{V_3}\left(\frac{\mu}{2}\right)}\right), \tag{B.75}$$

where all of the above constants come from Assumptions 4.7 and 4.10 the class-$\mathcal{K}_\infty$ functions come from Assumptions 4.4 and 4.7, and $L>0$ is such that $|\partial h_o/\partial\chi|\le L$ for all $\chi\in B_1$ with $B_1=\{\chi\in\mathbb{R}^q\mid|\chi|\le\Delta_1\}$ where $\Delta_1:=\Delta_\chi$. From the choice of $(\tilde{\Delta},\tilde{\Delta}_{u_1},\tilde{\Delta}_{u_2},\tilde{\mu})$, we generate $\overline{\varepsilon}^*$ such that Corollary 4.1 holds. Define $(\Delta_L,\Delta_{L_{u_1}},\Delta_{L_{u_2}},\Delta_{L_w})$ as $\Delta_L:=\Delta$, $\Delta_{L_{u_1}}:=\Delta_{u_1}$, $\Delta_{L_{u_2}}:=\Delta_{u_2}$, $\Delta_{L_w}:=\Delta_w$. Let Lemma 4.2 holds with the choice of $(\Delta_L,\Delta_{L_{u_1}},\Delta_{L_{u_2}},\Delta_{L_w})$, which means that Lemma 4.1 holds too. Introduce $\Delta_x:=\beta_{L_1}(\Delta,0)+\gamma_{L_1}(\Delta_{u_1})+\tilde{\gamma}_{L_1}(\Delta_{u_1})+\hat{\gamma}_{L_1}(\Delta_{u_2})+\mu_{L_1}$ where $\beta_{L_1}(\cdot,\cdot)$, $\gamma_{L_1}(\cdot)$, $\tilde{\gamma}_{L_1}(\cdot)$, $\hat{\gamma}_{L_1}(\cdot)$ and μ_{L_1} come from (4.17) in Lemma 4.1. We now define an auxiliary constant $\overline{C}:=\varepsilon c_1+c_2$ with

$$c_1 = (a_4+a_5+a_6+La_7)\alpha_{V_1}^2(\Delta_x)+\gamma_5^2(\Delta_{u_1})+\gamma_6^2(\Delta_{u_1}), \tag{B.76}$$

$$c_2 = \frac{[b_4 + b_5 + b_6 + Lb_7]^2}{\zeta_2}\alpha_W^2(\tilde{\mu}), \tag{B.77}$$

where all constants come from Assumption 4.7 and 4.10, $\tilde{\mu}$ is defined as in (B.75), $L > 0$ is as defined above, $\alpha_{V_1}(\cdot)$, $\alpha_W(\cdot)$, and $\gamma_i(\cdot)$ $(i = 5, 6)$ come from Assumptions 4.3, 4.4 and 4.10, respectively. Define the auxiliary constant

$$\hat{C} := \underline{\alpha}_{V_3}^{-1} \circ \overline{\alpha}_{V_3} \circ \tilde{\alpha}_{V_3}^{-1}\left(\frac{4}{\zeta_2}\overline{C}\right), \tag{B.78}$$

where it is observed that $\hat{C}$ can be made small by reducing $\overline{C}$, which implies to reduce ε and $\tilde{\mu}$. The weak triangle inequality for comparison functions leads to

$$\hat{C} \leq \underline{\alpha}_{V_3}^{-1} \circ \overline{\alpha}_{V_3} \circ \tilde{\alpha}_{V_3}^{-1}\left(\frac{8}{\zeta_2}\varepsilon c_1\right) + \underline{\alpha}_{V_3}^{-1} \circ \overline{\alpha}_{V_3} \circ \tilde{\alpha}_{V_3}^{-1}\left(\frac{8}{\zeta_2}c_2\right). \tag{B.79}$$

Note that $\tilde{\mu}$ in (B.75) is such that the second term on the right-hand side of (B.79) is half of μ, i.e., $\frac{1}{2}\mu = \underline{\alpha}_{V_3}^{-1} \circ \overline{\alpha}_{V_3} \circ \tilde{\alpha}_{V_3}^{-1}\left(\frac{8}{\zeta_2}c_2\right)$. So, let $(\Delta, \Delta_{u_1}, \Delta_{u_2}, \mu)$ generate

$$\varepsilon_a^* := \frac{\zeta_2\left[\tilde{\alpha}_{V_3}^{-1} \circ \overline{\alpha}_{V_3}^{-1} \circ \underline{\alpha}_{V_3}\left(\frac{1}{2}\mu\right)\right]}{8\left[(a_4 + a_5 + a_6 + La_7)\alpha_{V_1}^2(\Delta_x) + \gamma_5^2(\Delta_{u_1}) + \gamma_6^2(\Delta_{u_1})]\right]}, \tag{B.80}$$

such that $\mu > \hat{C}$ holds for all $\varepsilon \in (0, \varepsilon_a^*)$, which implies that $\frac{1}{2}\mu > \underline{\alpha}_{V_3}^{-1} \circ \overline{\alpha}_{V_3} \circ \tilde{\alpha}_{V_3}^{-1}\left(\frac{8}{\zeta_2}\varepsilon c_1\right)$. Hence, define

$$\varepsilon^* := \min\{\varepsilon_L^*, \overline{\varepsilon}^*, \varepsilon_a^*\}. \tag{B.81}$$

Note that ε_a^* is given by (B.80), $\overline{\varepsilon}^*$ is generated by Corollary 4.1, and ε_L^* in (B.81) comes from (B.59) in Lemma 4.2 and implies that Lemma 4.1 and Corollary 4.2 hold. It follows from Lemma 4.2 that

$$\begin{aligned}|e(t)| \leq \beta(|e_0|, t - t_0) + \underline{\alpha}_{V_3}^{-1} \circ \overline{\alpha}_{V_3} \circ \tilde{\alpha}_{V_3}^{-1}\Bigg(\frac{4}{\zeta_2}\Bigg[&\varepsilon(a_4 + a_5 + a_6 + La_7)\alpha_{V_1}^2(\|x[t_0, t]\|) \\ &+\varepsilon\gamma_5^2(\|u[t_0, t]\|) + \varepsilon\gamma_6^2(\|u[t_0, t]\|) + \frac{\overline{k}_1}{\zeta_2}\alpha_W^2(\|\xi[t_0, t]\|) + \gamma_{V_3}(\|w[t_0, t]\|)\Bigg]\Bigg),\end{aligned} \tag{B.82}$$

with $\overline{k}_1 = (b_4 + b_5 + b_6 + Lb_7)^2$. By virtue of Lemma 4.1 and Corollary 4.1, we have that $x(t)$ and $\xi(t)$ are bounded signal inputs to the error dynamics. We now use the cascade properties of the system to conclude the result. By using the ISS approach for interconnected systems proposed in [Lemma 4.7, 70], we have that (B.82) yield to

$$
\begin{aligned}
|e(t)| \le \beta_{T_1}\left(|(x_0,\xi_0,e_0)|, t-t_0\right) + \beta_e\left(2\underline{\alpha}_{V_3}^{-1}\circ\overline{\alpha}_{V_3}\circ\tilde{\alpha}_{V_3}^{-1}\left(\frac{4}{\zeta_2}(\varepsilon c_1 + c_2)\right), \frac{t-t_0}{2}\right)\\
+\gamma_{T_1}\left(|w\,[t_0,t]|\right) + \underline{\alpha}_{V_3}^{-1}\circ\overline{\alpha}_{V_3}\circ\tilde{\alpha}_{V_3}^{-1}\left(\frac{8}{\zeta_2}\varepsilon c_1\right) + \underline{\alpha}_{V_3}^{-1}\circ\overline{\alpha}_{V_3}\circ\tilde{\alpha}_{V_3}^{-1}\left(\frac{8}{\zeta_2}c_2\right), \qquad \text{(B.83)}
\end{aligned}
$$

where $\beta_{T_1}(\cdot,\cdot)\in\mathcal{KL}$ and $\gamma_{T_1}(\cdot)\in\mathcal{K}_\infty$ are given by (B.72) and (B.73), respectively. Therefore, by the fact that $\mu > \hat{C}$, $\beta(r,0) \ge \beta(r,s)$, and using (B.74), we conclude that (4.40) holds for all $\varepsilon\in(0,\varepsilon^*)$ and for all $|(x_0,\xi_0,\chi_0,e_0)| \le \Delta$, $\|u\|_\infty \le \Delta_{u_1}$, $\|\dot{u}\|_\infty \le \Delta_{u_2}$, $\|w\|_\infty \le \Delta_w$ and $t \ge t_0 > 0$.

Step 2) We now prove that (4.41) holds under Assumptions 4.1 - 4.10. Define the functions $\overline{\beta}_{T_1}(\cdot,\cdot)\in\mathcal{KL}$ and $\overline{\gamma}_{T_1}(\cdot)\mathcal{K}_\infty$

$$
\overline{\beta}_{T_1}(r,s) := \beta_e(r,s), \qquad \text{(B.84)}
$$

$$
\gamma_{T_1}(s) := \gamma_w(s), \qquad \text{(B.85)}
$$

where $\beta_e(\cdot,\cdot)\in\mathcal{KL}$ and $\gamma_w(\cdot)\in\mathcal{K}_\infty$ come from Lemma 4.2. For the given $\Delta > 0$, $\Delta_{u_1} > 0$, $\Delta_{u_2} > 0$, $\Delta_w > 0$ and $\mu > 0$, let $\varepsilon^* > 0$ be as defined in (B.81). As showed in Step 1), Lemmas 4.1 and 4.2 and Corollaries 4.1 and 4.2 hold. Using Corollary 4.1, let $(\tilde{\Delta},\tilde{\Delta}_{u_1},\tilde{\Delta}_{u_2},\tilde{\mu})$ generate $T^* > 0$ such that (4.20) in Corollary 4.1 holds for all $\varepsilon\in(0,\overline{\varepsilon}^*)$, $|(x_0,\xi_0)| \le \tilde{\Delta}$, $\|u\|_\infty \le \tilde{\Delta}_{u_1}$, $\|\dot{u}\|_\infty \le \tilde{\Delta}_{u_2}$, $t \ge \varepsilon T^* + t_0$. Hence, T^* is given by

$$
T^* := \varepsilon\frac{\upsilon - \nu}{K}, \qquad \text{(B.86)}
$$

where $\upsilon := \overline{\alpha}_W\circ\tilde{\alpha}_W^{-1}\left(\varepsilon\frac{4}{\zeta_3}[(b_2+b_3)^2\alpha_{V_1}^2(\tilde{\Delta}_x) + \gamma_2^2(\tilde{\Delta}_{u_1}) + \gamma_3^2(\tilde{\Delta}_{u_1}) + \gamma_4^2(\tilde{\Delta}_{u_2})]\right)$ with $\tilde{\Delta}_x = \Delta_x$, $\tilde{\alpha}_W(\cdot) := \alpha_W^2(\cdot)$, $\nu := \overline{\alpha}_W(\tilde{\Delta})$, $K = \min\{\hat{\alpha}_W(|\xi|)\}$ over the set $\{\tilde{\Delta} \le |\xi| \le \tilde{\Delta}_\xi\}$ with $\hat{\alpha}_W(\cdot) = \frac{\zeta_3}{4}\alpha_W^2\circ\overline{\alpha}_W^{-1}(\cdot)$ and $\tilde{\Delta}_\xi = \underline{\alpha}_W^{-1}\circ\overline{\alpha}_W(\tilde{\Delta})$, where all the constants and class-$\mathcal{K}_\infty$ functions come from Assumptions 3, 4 and 5.

To show the result, we consider (B.82) which comes from Lemma 4.2. It is observed that $x(t)$ and $\xi(t)$ are essentially bounded signal inputs to the error dynamics. We know from the SPA result in Corollary 4.1 that the fast state rapidly converges, and it becomes ultimately bounded by $\tilde{\mu}$ after a finite time $T^* > 0$ defined by (B.86). This occurs because $\beta_\xi(\cdot,\cdot)\in\mathcal{KL}$ in (4.19) quickly converges to zero. Hence, $|\xi[\varepsilon T^*, t]| \le \tilde{\mu}$ for all $t \ge \varepsilon T^* + t_0$ where $\tilde{\mu}$ is given by (B.75). Moreover, $|x[t_0,t]| \le \Delta_x$, and $|u[t_0,t]| \le \Delta_{u_1}$. Therefore, by considering c_1 and c_2 in (B.76) and (B.77), (B.82) leads to

$$\begin{aligned}|e(t)| \leq\ & \overline{\beta}_{T_1}(|e(t_0)|, t-t_0) + \underline{\alpha}_{V_3}^{-1} \circ \overline{\alpha}_{V_3} \circ \tilde{\alpha}_{V_3}^{-1}\left(\frac{8}{\zeta_2}\varepsilon c_1\right) \\ & + \underline{\alpha}_{V_3}^{-1} \circ \overline{\alpha}_{V_3} \circ \tilde{\alpha}_{V_3}^{-1}\left(\frac{8}{\zeta_2}c_2\right) + \overline{\gamma}_{T_1}(|w[t_0, t]|),\end{aligned} \tag{B.87}$$

for all $\varepsilon \in (0, \varepsilon^*)$ and for all $t \geq \varepsilon T^* + t_0$, where $\overline{\beta}_{T_1}(\cdot,\cdot) \in \mathcal{K}_\infty$ and $\overline{\gamma}_{T_1}(\cdot) \in \mathcal{K}_\infty$ given by (B.84) and (B.85). Note that the sum of second and third terms on the right-hand side of (B.87) is equal to right hand side of (B.79), which is smaller than μ for all $\varepsilon \in (0, \varepsilon^*)$. Therefore, we conclude that (4.41) holds for all $\varepsilon \in (0, \varepsilon^*)$, $|(x_0, \xi_0, \chi_0, e_0)| \leq \Delta$, $\|u\|_\infty \leq \Delta_{u_1}$, $\|\dot{u}\|_\infty \leq \Delta_{u_2}$, $\|w\|_\infty \leq \Delta_w$ and $t \geq \varepsilon T^* + t_0$.

Step 3) We now prove the $\mathcal{L}_2$ stability property (4.42). Define

$$k_{T_1} := \frac{2}{\zeta_2}, \tag{B.88}$$

and the class-$\mathcal{K}_\infty$ functions

$$\alpha_{T_1}(s) := k_{T_1}\overline{\alpha}_{V_3}(s), \tag{B.89}$$

$$\overline{\alpha}_{T_1}(s) := \frac{k_{T_1}}{\zeta_2}(b_4 + b_5 + b_6 + Lb_7)^2 \alpha_{W_c}(s), \tag{B.90}$$

where $\alpha_{W_c} \in \mathcal{K}_\infty$ come from Corollary 4.1. For the given $\Delta > 0$, $\Delta_{u_1} > 0$, $\Delta_{u_2} > 0$, $\Delta_w > 0$ and $\mu > 0$, let $|(x_0, \xi_0, \chi_0, e_0)| \leq \Delta$, $\|u\|_{\mathcal{L}_2} \leq \Delta_{u_1}$, $\|\dot{u}\|_{\mathcal{L}_2} \leq \Delta_{u_2}$, $\|w\|_{\mathcal{L}_2} \leq \Delta_w$. Let Lemmas 4.1 and 4.2 and Corollary 4.2 hold as in Step 1) of this proof where the $\mathcal{L}_2$ results hold for any input satisfying $\|u\|_\infty \leq \Delta_{u_1}$, $\|\dot{u}\|_\infty \leq \Delta_{u_2}$, $\|u\|_{\mathcal{L}_2} \leq \Delta_{u_1}$, $\|\dot{u}\|_{\mathcal{L}_2} \leq \Delta_{u_2}$, and for any $\|w\|_\infty \leq \Delta_w$, $\|w\|_{\mathcal{L}_2} \leq \Delta_w$. Define $(\tilde{\Delta}, \tilde{\Delta}_{u_1}, \Delta_{u_2}, \tilde{\mu})$ as $\tilde{\Delta} := \Delta$, $\tilde{\Delta}_{u_1} = \Delta_{u_1}$, $\tilde{\Delta}_{u_2} = \Delta_{u_2}$ and

$$\tilde{\mu} := \alpha_W^{-1}\left(\sqrt{\frac{\zeta_2^2}{4(b_4 + b_5 + b_6 + Lb_7)^2}\mu}\right). \tag{B.91}$$

From the choice of $(\tilde{\Delta}, \tilde{\Delta}_{u_1}, \Delta_{u_2}, \tilde{\mu})$, we generate $\overline{\varepsilon}^* > 0$ such that Corollary 4.1 holds. Define

$$\mu_{\mathcal{L}_2} := \frac{k_{T_1}}{\zeta_2}(b_4 + b_5 + b_6 + Lb_7)^2 \alpha_W^{-1}\left(\sqrt{\frac{\zeta_2^2}{4(b_4 + b_5 + b_6 + Lb_7)^2}\mu}\right), \tag{B.92}$$

where $\alpha_w(\cdot) \in \mathcal{K}_\infty$ and the rest of the constant come from Assumptions 4.4, 4.5 and 4.10, and L is defined as in Step 1) of this proof. Let $\varepsilon_b^* > 0$ be such that

$$\varepsilon \frac{2}{\zeta_2}\left[(a_4 + a_5 + a_6 + La_7)\alpha_{V_1}^2(\Delta_x) + \gamma_5^2(\Delta_{u_1}) + \gamma_6^2(\Delta_{u_1})\right] \leq \frac{1}{2}\mu,$$

for all $\varepsilon \in (0, \varepsilon_b^*)$. Hence, define

$$\varepsilon_b^* := \frac{\zeta_2\mu}{4\left[(a_4 + a_5 + a_6 + La_7)\alpha_{V_1}^2(\Delta_x) + \gamma_5^2(\Delta_{u_1}) + \gamma_6^2(\Delta_{u_2})\right]}, \tag{B.93}$$

and

$$\varepsilon_{\mathcal{L}_2}^* := \min\{\varepsilon_L^*, \overline{\varepsilon}^*, \varepsilon_b^*\}, \tag{B.94}$$

where ε_L^* and $\overline{\varepsilon}^*$ come from Lemma 4.2 and Corollary 4.1, respectively. We now consider (4.39) in Lemma 4.2 and (4.21) in Corollary 4.1. Note that (4.39) holds for all $\varepsilon \in (0, \varepsilon^*)$ since $\varepsilon_L^* \leq \varepsilon^*$. It follows that (4.39) can be written as follows

$$\begin{aligned}\int_{t_0}^t \alpha_{V_3}^2(|e(\tau)|)d\tau \leq{}& \frac{2}{\zeta_2}\overline{\alpha}_{V_3}(|e_0|) + \varepsilon\frac{2}{\zeta_2}(a_4 + a_5 + a_6 + La_7)\int_{t_0}^t \alpha_{V_1}^2(|x(\tau)|)d\tau \\ &+ \varepsilon\frac{2}{\zeta_2}\int_{t_0}^t \gamma_5^2(|u(\tau)|)d\tau + \varepsilon\frac{2}{\zeta_2}\int_{t_0}^t \gamma_6^2(|u(\tau)|)d\tau + \frac{2}{\zeta_2}\int_{t_0}^t \gamma_{V_3}(|w(\tau)|)d\tau \\ &+ \frac{2\overline{k}_1}{\zeta_2^2}\left(\varepsilon\alpha_{W_c}(|\xi_0|) + \tilde{\mu}(t - t_0)\right),\end{aligned} \tag{B.95}$$

where $\alpha_{W_c}(\cdot)$ comes from Corollary 4.1 and $\overline{k}_1 = (b_4 + b_5 + b_6 + Lb_7)^2$. We use the fact that $|x(t)| \leq \Delta_x$ and $|u(t)| \leq \Delta_{u_1}$ to obtain

$$\begin{aligned}\int_{t_0}^t \alpha_{V_3}^2(|e(\tau)|)d\tau \leq{}& \frac{2}{\zeta_2}\overline{\alpha}_{V_3}(|e_0|) + \frac{2\overline{k}_1}{\zeta_2^2}\left(\varepsilon\alpha_{W_c}(|\xi_0|) + \tilde{\mu}(t - t_0)\right) + \frac{2}{\zeta_2}\int_{t_0}^t \gamma_{V_3}(|w(\tau)|)d\tau \\ &+ \varepsilon\frac{2}{\zeta_2}\left[(a_4 + a_5 + a_6 + La_7)\alpha_{V_1}^2(\Delta_x) + \gamma_5^2(\Delta_{u_1}) + \gamma_6^2(\Delta_{u_1})\right]\int_{t_0}^t d\tau,\end{aligned} \tag{B.96}$$

Eq. (B.96) and $\varepsilon\frac{2}{\zeta_2}\left[(a_4 + a_5 + a_6 + La_7)\alpha_{V_1}^2(\Delta_x) + \gamma_5^2(\Delta_{u_1}) + \gamma_6^2(\Delta_{u_1})\right] \leq \frac{1}{2}\mu$ lead to

$$\begin{aligned}\int_{t_0}^t \alpha_{V_3}^2(|e(\tau)|)d\tau \leq{}& \frac{2}{\zeta_2}\overline{\alpha}_{V_3}(|e_0|) + \varepsilon\frac{2\overline{k}_1}{\zeta_2^2}\alpha_{W_c}(|\xi_0|) + \frac{2}{\zeta_2}\int_{t_0}^t \gamma_{V_3}(|w(\tau)|)d\tau \\ &+ \frac{2\overline{k}_1}{\zeta_2^2}\tilde{\mu}(t - t_0) + \mu(t - t_0),\end{aligned} \tag{B.97}$$

for all $\varepsilon \in (0, \varepsilon^*_{\mathcal{L}_2})$. By using $\alpha_{T_1}(\cdot), \overline{\alpha}_{T_1}(\cdot) \in \mathcal{K}_\infty$, k_{T_1} and $\mu_{\mathcal{L}_2}$ given in (B.88) - (B.92) and by the fact that $|e_0| \leq |(x_0, \xi_0, e_0)|$ and $|\xi_0| \leq |(x_0, \xi_0, e_0)|$, we conclude from (B.97) that (4.42) holds for all $\varepsilon \in (0, \varepsilon^*_{\mathcal{L}_2})$, $|(x_0, \xi_0, \chi_0, e_0)| \leq \Delta$, for any input satisfying $||u||_\infty \leq \Delta_{u_1}$, $||\dot{u}||_\infty \leq \Delta_{u_2}$, $||u||_{\mathcal{L}_2} \leq \Delta_{u_1}$, $||\dot{u}||_{\mathcal{L}_2} \leq \Delta_{u_2}$, for any $||w||_\infty \leq \Delta_w$, $||w||_{\mathcal{L}_2} \leq \Delta_w$, and for all $t \geq t_0 \geq 0$.

Step 4) Finally, we demonstrate that the $\mathcal{L}_2$ stability property (4.43). To prove this final result, let Lemmas 4.1 and 4.2 and Corollary 4.2 hold as in previous steps, and let Corollary 4.1 holds as stated in Step 3) of this proof. Let $k_{T_1} > 0$ and $\alpha_{T_1}(\cdot) \in \mathcal{K}_\infty$ be such as defined in (B.88) and (B.89), respectively. Moreover, define $\varepsilon^*_{\mathcal{L}_2} > 0$ as in (B.94). From the choice of $(\tilde{\Delta}, \tilde{\Delta}_{u_1}, \Delta_{u_2}, \tilde{\mu})$ in Step 3) where $\tilde{\mu}$ is given by (B.91), we generate $T^* > 0$ such that $|\xi(t)| \leq \tilde{\mu}$ for all $t \geq \varepsilon T^* + t_0$. Since we work with the intersection of the infinity norm and the $\mathcal{L}_2$ norm, we use the fact that $|\xi(t)| \leq \tilde{\mu}$ for all $t \geq \varepsilon T^* + t_0$ and we consider $|x(t)| \leq \Delta_x$, $|u(t)| \leq \Delta_{u_1}$ and $|\dot{u}(t)| \leq \Delta_{u_2}$. Therefore, we obtain from (4.39) that the following holds

$$\begin{aligned}\int_{t_0}^{t} \alpha^2_{V_3}(|e(\tau)|)d\tau \leq \frac{2}{\zeta_2}\overline{\alpha}_{V_3}(|e_0|) + \frac{2\overline{k}_1}{\zeta_2^2}\alpha^2_W(\tilde{\mu})\int_{t_0}^{t} d\tau + \varepsilon\frac{2}{\zeta_2}\Big[(a_4 + a_5 + a_6 + La_7)\alpha^2_{V_1}(\Delta_x) \\ + \Big[\gamma_5^2(\Delta_{u_1}) + \gamma_6^2(\Delta_{u_1})\Big]\Big]\int_{t_0}^{t} d\tau + \frac{2}{\zeta_2}\int_{t_0}^{t}\gamma_{V_3}(|w(\tau)|)d\tau,\end{aligned} \tag{B.98}$$

for all $t \geq \varepsilon T^* + t_0$. From the choice of $\tilde{\mu}$ in (B.91), we have that $\frac{2\overline{k}_1}{\zeta_2^2}\alpha^2_W(\tilde{\mu}) \leq \frac{1}{2}\mu$. Moreover, it follows from Step 3) of this proof that

$$\varepsilon\frac{2}{\zeta_2}\Big[(a_4 + a_5 + a_6 + La_7)\alpha^2_{V_1}(\Delta_x) + \Big[\gamma_5^2(\Delta_{u_1}) + \gamma_6^2(\Delta_{u_1})\Big]\Big] \leq \frac{1}{2}\mu, \tag{B.99}$$

for all $\varepsilon \in (0, \varepsilon^*_b)$ where $\varepsilon^*_b \leq \varepsilon^*_{\mathcal{L}_2}$. Hence, (B.98) leads to

$$\int_{t_0}^{t} \alpha^2_{V_3}(|e(\tau)|)d\tau \leq \frac{2}{\zeta_2}\overline{\alpha}_{V_3}(|e_0|) + \frac{2}{\zeta_2}\int_{t_0}^{t}\gamma_{V_3}(|w(\tau)|)d\tau + \mu(t - t_0). \tag{B.100}$$

Therefore, by using $k_{T_1} > 0$ as in (B.88) and $\alpha_{T_1}(\cdot) \in \mathcal{K}_\infty$ as in (B.89), it follows from (B.100) that (4.43) holds for all $\varepsilon \in (0, \varepsilon^*_{\mathcal{L}_2})$, $|(x_0, \xi_0, \chi_0, e_0)| \leq \Delta$, for any input satisfying $||u||_\infty \leq \Delta_{u_1}$, $||\dot{u}||_\infty \leq \Delta_{u_2}$, $||u||_{\mathcal{L}_2} \leq \Delta_{u_1}$, $||\dot{u}||_{\mathcal{L}_2} \leq \Delta_{u_2}$, for any $||w||_\infty \leq \Delta_w$, $||w||_{\mathcal{L}_2} \leq \Delta_w$, and for all $t \geq \varepsilon T^* t_0$. This completes the proof. ■

B.6 Proof of Lemma B.1

We split the proof into two steps. In the first step, we show that the infimum always happens at the boundary, i.e. $|(r,l)| = s$. Then, in the second step, we show that the function is strictly increasing function and it is zero at $|(r,l)| = 0$.

Step 1) We use contradiction by assuming that, for a given s_1, the infimum happens at $|(r_2,l_2)| = \underbrace{s_1 + \nu}_{s_2}$ for some $\nu > 0$. This means that

$$\kappa_1(|r_2|) + \kappa_2(|l_2|) \leq \kappa_1(|r_1|) + \kappa_2(|l_1|). \tag{B.101}$$

for any (r_1,l_1) in the neighbourhood of (r_2,l_2). Now, chose the pair (r_1,l_1) such that $|r_1| \leq |r_2|$, $|l_1| < |l_2|$ and $|(r_1,l_1)| = s_1$. Since $\kappa_1(\cdot)$ and $\kappa_2(\cdot)$ are strictly increasing functions, it follows that

$$\kappa_1(|r_1|) + \kappa_2(|l_1|) < \kappa_1(|r_2|) + \kappa_2(|l_2|). \tag{B.102}$$

This contradicts with the assumption that infimum happens at $|(r_2,l_2)|$.

Step 2) Again, we use contradiction by assuming if $s_1 < s_2$ then $\hat{\kappa}(s_1) \geq \hat{\kappa}(s_2)$. Let (r_1,l_1) and (r_2,l_2) be the points at which infimums happen. Following the same argument in the first step, we can find a point (r_1^*,l_1^*) in the neighbourhood of (r_2,l_2) such that

$$\kappa_1(|r_1^*|) + \kappa_2(|l_1^*|) < \kappa_1(|r_2|) + \kappa_2(|l_2|) \leq \hat{\kappa}(s_1), \tag{B.103}$$

with $|(r_1^*,l_1^*)| = s_1$. According to definition of $\hat{\kappa}(s_1)$, we have $\hat{\kappa}(s_1) \leq \kappa_1(|r_1^*|) + \kappa_2(|l_1^*|)$ which contradicts with the inequality in (B.103). Therefore, $\hat{\kappa}(s)$ is strictly increasing. Since $\kappa_1(\cdot)$ and $\kappa_2(\cdot)$ are of class-$\mathcal{K}_\infty$, it is straightforward to see that $\hat{\kappa}(s) = 0$ and $\lim_{s\to\infty} \hat{\kappa}(s) = \infty$ if and only if $(r,l) = 0$ and $|(r,l)| \to \infty$ respectively. This completes the proof. ■

B.7 Proof of Lemma B.2

Let define variables $a = (1-d)\left(\zeta_1 - \varepsilon\left(a_1 + \frac{1}{4}\right)\right)$, $b = \frac{1}{\varepsilon}d\left(\frac{2}{3}\zeta_3 - \varepsilon(a_2+a_3)\right)$ and $c = -\frac{1}{2}(1-d)b_1 - \frac{1}{2}d(b_2+b_3)$. Note that the statement of the Lemma implies that d is fixed. Then, it can be observed that the minimum eigenvalue of the matrix $\mathbf{A}(d,\varepsilon)$ in (B.15) is a function of ε and is expressed by

$$\lambda_{\min}(\varepsilon) = \frac{a + b - \sqrt{(a+b)^2 - 4(ab - c^2)}}{2}, \tag{B.104}$$

for all $\varepsilon > 0$. From the continuity of eigenvalues with respect to the parameter ε, we have $\lim_{\varepsilon \to 0} \lambda_{\min}(\varepsilon) = (1-d)\zeta_1$. It is also possible to show that $\lambda_{\min}(\varepsilon) = 0$ for $\underline{\varepsilon}^*$ given by

$$\underline{\varepsilon}^* := \frac{\tilde{b} - \sqrt{\tilde{b}^2 - \frac{8}{3}\zeta_1\zeta_3(a_2 + a_3)\left(a_1 + \frac{1}{4}\right)}}{2(a_2 + a_3)\left(a_1 + \frac{1}{4}\right)}, \tag{B.105}$$

with

$$\tilde{b} = \frac{1}{4d(1-d)}((1-d)b_1 + d(b_2 + b_3))^2 + \zeta_1(a_2 + a_3) + \frac{2}{3}\zeta_3\left(a_1 + \frac{1}{4}\right).$$

Observe that it is straightforward to prove that the term inside the square root is always positive.

Now, we claim that $\frac{\mathbf{d}\lambda_{\min}(\varepsilon)}{\mathbf{d}\varepsilon} < 0$, which along with continuity of eigenvalues with respect to the parameter ε conclude that $\lambda_{\min}(\varepsilon)$ is strictly decreasing function of ε. To do so, we use contradiction.
Assume now that $\lambda_{\min}(\varepsilon)$ is not strictly decreasing. Then, it means that $\frac{\mathbf{d}\lambda_{\min}(\varepsilon)}{\mathbf{d}\varepsilon} = 0$ for some values of ε, which are obtained from the following equation,

$$\frac{\mathbf{d}}{\mathbf{d}\varepsilon}[a+b] - \frac{\left(\frac{\mathbf{d}}{\mathbf{d}\varepsilon}[a-b]\right)(a-b)}{\sqrt{(a-b)^2 + 4c^2}} = 0, \tag{B.106}$$

or equivalently

$$\begin{aligned} \left(\frac{\mathbf{d}}{\mathbf{d}\varepsilon}[a+b]\right)^2 \left((a-b)^2 + 4c^2\right) &= \left(\frac{\mathbf{d}}{\mathbf{d}\varepsilon}[a-b]\right)^2 (a-b)^2 \\ \Longleftrightarrow \left(\frac{\mathbf{d}}{\mathbf{d}\varepsilon}[a+b]\right)^2 \left(4c^2\right) &= \underbrace{\left(\left(\frac{\mathbf{d}}{\mathbf{d}\varepsilon}[a-b]\right)^2 - \left(\frac{\mathbf{d}}{\mathbf{d}\varepsilon}[a+b]\right)^2\right)}_{(**)}(a-b)^2. \end{aligned} \tag{B.107}$$

Now, the term, indicated by $(**)$ in (B.107), is equal to

$$\left(\left(\frac{\mathbf{d}}{\mathbf{d}\varepsilon}[a-b]\right)^2 - \left(\frac{\mathbf{d}}{\mathbf{d}\varepsilon}[a+b]\right)^2\right) = \left(\frac{\mathbf{d}}{\mathbf{d}\varepsilon}[a-b] - \frac{\mathbf{d}}{\mathbf{d}\varepsilon}[a+b]\right)\left(\frac{\mathbf{d}}{\mathbf{d}\varepsilon}[a-b] + \frac{\mathbf{d}}{\mathbf{d}\varepsilon}[a+b]\right)$$

$$
\begin{aligned}
&= -4\left(\frac{\mathbf{d}}{\mathbf{d}\varepsilon}[b]\right)\left(\frac{\mathbf{d}}{\mathbf{d}\varepsilon}[a]\right) \\
&= -4\left(-\frac{\frac{2}{3}d\zeta_3}{\varepsilon^2}\right)\left(-(1-d)(\gamma_1+\frac{1}{4})\right),
\end{aligned} \tag{B.108}
$$

which is negative; whilst the left term in (B.107) and $(a-b)^2$ are positive. This means that there is no ε that satisfies (B.107), and subsequently, (B.106). This contradicts with our assumption and completes the proof.

Remark B.1. *Consider $\tilde{\varepsilon}^*$ and $\underline{\varepsilon}^*$ given in (B.5) and (B.105), respectively. Note that at* $d = d^*$ *the inequality $\tilde{\varepsilon}^* < \underline{\varepsilon}^*$ holds. To verify so, observe that $\tilde{\varepsilon}^* < \underline{\varepsilon}^*$ implies*

$$
\frac{\hat{b} - \sqrt{\hat{b}^2 - \frac{8}{3}\zeta_1\zeta_3\left(a_1+\frac{1}{4}\right)(a_2+a_3)}}{2\left(a_1+\frac{1}{4}\right)(a_2+a_3)} > \frac{\frac{2}{3}\zeta_1\zeta_3}{\hat{b}}
$$

with $\hat{b} = b_1(b_2+b_3) + \zeta_1(a_2+a_3) + \frac{2}{3}\zeta_3\left(a_1+\frac{1}{4}\right)$. *So, we have that*

$$
\begin{aligned}
&\hat{b}^2 - \frac{4}{3}\zeta_1\zeta_3\left(a_1+\frac{1}{4}\right)(a_2+a_3) > \hat{b}\sqrt{\hat{b}^2 - \frac{8}{3}\zeta_1\zeta_3\left(a_1+\frac{1}{4}\right)(a_2+a_3)}, \\
&\hat{b}^4 + \frac{16}{9}(\zeta_1\zeta_3)^2\left(a_1+\frac{1}{4}\right)^2(a_2+a_3)^2 \\
&\quad -\frac{8}{3}\zeta_1\zeta_3\left(a_1+\frac{1}{4}\right)(a_2+a_3)\hat{b}^2 > \hat{b}^4 - \frac{8}{3}\zeta_1\zeta_3\left(a_1+\frac{1}{4}\right)(a_2+a_3)\hat{b}^2, \\
&\frac{16}{9}(\zeta_1\zeta_3)^2\left(a_1+\frac{1}{4}\right)^2(a_2+a_3)^2 > 0.
\end{aligned}
$$

Since the last inequality is true, it follows that, in fact, $\tilde{\varepsilon}^* < \underline{\varepsilon}^*$. Finally, the monotonicity of $\lambda_{\min}\{\mathbf{A}(d^*,\varepsilon)\}$ *and the fact that $\tilde{\varepsilon}^* < \underline{\varepsilon}^*$ imply*

$$
\lambda_{\min}\{\mathbf{A}(d^*,\tilde{\varepsilon}^*)\} > \lambda_{\min}\{\mathbf{A}(d^*,\underline{\varepsilon}^*)\}.
$$

From Lemma B.2, we have that $\lim_{\varepsilon\to\underline{\varepsilon}^*}\lambda_{\min}\{\mathbf{A}(d^*,\varepsilon)\} = 0$. *Therefore,* $\lambda_{\min}\{\mathbf{A}(d^*,\tilde{\varepsilon}^*)\} > 0$.

Appendix C
Proofs of Chapter 6

C.1 Proof of Lemma 6.3

Let $\hat{x}_i \in \mathbf{X}$, for $i \in \{1, \ldots, N\}$, and let Assumption 6.6 hold such that we generate $V_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$, $\tilde{\gamma} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}_{\geq 0}$, $a_i > 0$ for $i \in \{1, \ldots, 4\}$, and $\lambda_0 > 0$. Define $k_{L1} > 0$, $\lambda_{L1} > 0$ and $\tilde{\varepsilon}^* > 0$ as follows

$$k_{L1} := \sqrt{\frac{a_2}{a_1}}, \tag{C.1}$$

$$\lambda_{L1} := \frac{1}{2}(1-\theta)\lambda_0, \tag{C.2}$$

$$\tilde{\varepsilon}^* := \frac{a_1}{a_4 L_x}\theta\lambda_0, \tag{C.3}$$

where $\theta \in (0,1)$ and $L_x > 0$ is such that $|x(t)|_\infty \leq \varepsilon L_x$. Let $\overline{\Delta} > 0$, $\overline{\Delta}_{e_\xi} > 0$ and $\overline{\Delta}_{u_1} > 0$ be given such that $|\xi(0)|_\infty \leq \overline{\Delta}$, $|e_{\xi_i}(0)|_\infty \leq \overline{\Delta}_{e_\xi}$, $\|u\|_\infty \leq \overline{\Delta}_{u_1}$. To apply Assumption 6.2, define (Δ, Δ_{u_1}) as $\Delta := \overline{\Delta}$ and $\Delta_{u_1} := \overline{\Delta}_{u_1}$. From the definition of (Δ, Δ_{u_1}), we have that $k_{A2} > 0$ is generated by Assumption 6.2. Let us introduce

$$\hat{\gamma}(e_{x_i}) = \max_{|\xi|_\infty \leq k_{A2}, |u|_\infty \leq \overline{\Delta}_{u_1}} \tilde{\gamma}(e_{x_i}, \xi, u). \tag{C.4}$$

It follows from [Lemma 4.3, 70] that there exists $\overline{\gamma}(\cdot) \in \mathcal{K}_\infty$ such that

$$\overline{\gamma}(s) \geq \max_{s \geq |e_{x_i}|_\infty} \hat{\gamma}(e_{x_i}), \tag{C.5}$$

for $s \geq 0$. Then, define the class-$\mathcal{K}_\infty$ function

$$\gamma_L(s) := \sqrt{\frac{1}{(1-\theta)a_1\lambda_0}\overline{\gamma}(s)}. \tag{C.6}$$

We now prove that the conclusion of the lemma holds. We use $V_i(x, e_{\xi_i})$ satisfying Assumption 6.6, for $i \in \{1, \ldots, N\}$, as Lyapunov function candidates for the error systems (6.34). We have that time derivatives of $V_i(x, e_{\xi_i})$, for $i \in \{1, \ldots, N\}$, along the trajectories of (6.1) and (6.34) are given by

$$\dot{V}_i(x, e_{\xi_i}) = \frac{\partial V_i}{\partial x}\dot{x} + \frac{\partial V_i}{\partial e_{\xi_i}}\bar{f}_{e_i}(\xi, x, e_{\xi_i}, e_{x_i}, u), \tag{C.7}$$

for all $t \geq 0$. It follows from (C.7) that

$$\dot{V}_i(x, e_{\xi_i}) \leq -\lambda_0 V_i(x, e_{\xi_i}) + \tilde{\gamma}(e_{x_i}, \xi, u) + a_4 |e_{\xi_i}|_\infty^2 |\dot{x}|_\infty. \tag{C.8}$$

Since $\tilde{\gamma} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}_{\geq 0}$ is a continuous function and $\tilde{\gamma}(0, \xi, u) = 0$, for all $\xi \in \mathbb{R}^m$, $u \in \mathbb{R}^r$, $\tilde{\gamma}$ can always be upper bounded by the positive definite function in (C.4). Hence, by using that $|\dot{x}|_\infty \leq \varepsilon L_x$ and (C.4), we conclude from (C.8) that

$$\dot{V}_i(x, e_{\xi_i}) \leq -\lambda_0 V_i(x, e_{\xi_i}) + \hat{\gamma}(e_{x_i}) + \varepsilon a_4 L_x |e_{\xi_i}|_\infty^2. \tag{C.9}$$

Then, it follows from (6.35), (C.5) and (C.9) that

$$\dot{V}_i(x, e_{\xi_i}) \leq -\left[\lambda_0 - \frac{a_4 L_x}{a_1}\varepsilon\right] V_i(x, e_{\xi_i}) + \overline{\gamma}(e_{x_i}). \tag{C.10}$$

Let consider $\theta \in (0, 1)$ such that (C.10) is written as follows

$$\dot{V}_i(x, e_{\xi_i}) \leq -(1-\theta)\lambda_0 V_i(x, e_{\xi_i}) - \theta\lambda_0 V_i(x, e_{\xi_i}) + \frac{a_4 L_x}{a_1}\varepsilon V_i(x, e_{\xi_i}) + \overline{\gamma}(e_{x_i}). \tag{C.11}$$

Then, we conclude that, for any $\varepsilon \in (0, \tilde{\varepsilon}^*)$ with $\tilde{\varepsilon}^*$ given by (C.3), the time derivative of $V_i(x, e_{\xi_i})$ is bounded as follows

$$\dot{V}_i(x, e_{\xi_i}) \leq -(1-\theta)\lambda_0 V_i(x, e_{\xi_i}) + \overline{\gamma}(e_{x_i}). \tag{C.12}$$

By the comparison principle in [Lemma 3.4, 70], for any $|\xi(0)|_\infty \leq \overline{\Delta}$, $|e_{\xi_i}(0)|_\infty \leq \overline{\Delta}_{e_\xi}$, $\|u\|_\infty \leq \overline{\Delta}_{u_1}$, it follows from (C.12) that the corresponding solutions to (6.1) and (6.34) verify

$$\begin{aligned} V_i(x, e_{\xi_i}) \leq{}& \exp\left[-(1-\theta)\lambda_0 t\right] V_i(x(0), e_{\xi_i}(0)) \\ &+ \overline{\gamma}(\|e_{x_i}\|_\infty) \int_0^t \exp\left[-(1-\theta)\lambda_0 \tau\right] d\tau. \end{aligned} \tag{C.13}$$

Then, for any $\varepsilon \in (0, \tilde{\varepsilon}^*)$, we have from (C.13) that

$$|e_{\xi_i}(t)|_\infty \le \sqrt{\frac{a_2}{a_1}\exp\left(-\bar{\lambda}_0 t\right)}|e_{\xi_i}(0)|_\infty + \sqrt{\frac{1}{a_1}\bar{\gamma}(\|e_{x_i}\|_\infty)\int_0^t \exp\left(-\bar{\lambda}_0\tau\right) d\tau}, \tag{C.14}$$

where $\bar{\lambda}_0 = (1-\theta)\lambda_0$ and where we have used (6.35) and that $\sqrt{\mathbf{a}+\mathbf{b}} \le \sqrt{\mathbf{a}} + \sqrt{\mathbf{b}}$ for any $\mathbf{a} \ge 0$, $\mathbf{b} \ge 0$. It follows from (C.14) that

$$|e_{\xi_i}(t)|_\infty \le \sqrt{\frac{a_2}{a_1}}\exp\left(-\frac{1}{2}\bar{\lambda}_0 t\right)|e_{\xi_i}(0)|_\infty + \sqrt{\frac{1}{a_1\bar{\lambda}_0}\left[1-\exp\left(-\bar{\lambda}_0 t\right)\right]\bar{\gamma}(\|e_{x_i}\|_\infty)}. \tag{C.15}$$

Since $1-\exp\left(-\bar{\lambda}_0 t\right) \le 1$, we obtain from (C.15) that

$$|e_{\xi_i}(t)|_\infty \le \sqrt{\frac{a_2}{a_1}}\exp\left(-\frac{1}{2}\bar{\lambda}_0 t\right)|e_{\xi_i}(0)|_\infty + \sqrt{\frac{1}{a_1\bar{\lambda}_0}\bar{\gamma}(\|e_{x_i}\|_\infty)}, \tag{C.16}$$

for $\varepsilon \in (0, \tilde{\varepsilon}^*)$ and $t \ge 0$. Therefore, by using the definitions given in (C.1), (C.2) and (C.6), we conclude that (6.41) holds. This completes the proof. ■

C.2 Proof of Lemma 6.4

Let $\underline{\Delta} > 0$, $\underline{\Delta}_{e_\xi} > 0$, $\underline{\Delta}_{u_1} > 0$ and $\underline{\nu} > 0$ be given such that $|\xi(0)|_\infty \le \underline{\Delta}$, $|e_{\xi_i}(0)|_\infty \le \underline{\Delta}_{e_\xi}$, $\|u\|_\infty \le \underline{\Delta}_{u_1}$. To apply Assumption 6.2, define (Δ, Δ_{u_1}) as $\Delta := \underline{\Delta}$ and $\Delta_{u_1} := \underline{\Delta}_{u_1}$. From the definition of (Δ, Δ_{u_1}), we have that $k_{A2} > 0$ is generated by Assumption 6.2. To apply Assumption 6.7, define $(\Delta, \Delta_{e_\xi}, \Delta_{u_1})$ as $\Delta := \underline{\Delta}$, $\Delta_{e_\xi} := \underline{\Delta}_{e_\xi}$ and $\Delta_{u_1} := \underline{\Delta}_{u_1}$. Hence, from the definition of $(\Delta, \Delta_{e_\xi}, \Delta_{u_1})$, we use Assumption 6.7 to generate $\alpha_{A7}(\cdot) \in \mathcal{KL}$ and $T_{A7} > 0$. Define

$$\alpha_L(s) := \frac{1}{2p}\alpha_{A7}(s). \tag{C.17}$$

Since $x(t) \in \mathbf{X}$ where $\mathbf{X}$ is assumed to be a known compact set, it follows that $\hat{x}_i - x(t)$ belongs to some compact set $\tilde{\mathbf{X}}$. Therefore, there is $k_{e_x} > 0$ such that

$$|\hat{x}_i - x(t)|_\infty \le k_{e_x}, \tag{C.18}$$

for all $i \in \{1 \ldots, N\}$ and $t \geq 0$ where $N \in \mathbb{N}_{\geq 1}$. Note that (C.18) implies $|e_{x_i}(t)|_\infty \leq k_{e_x}$ for all $i \in \{1 \ldots, N\}$ and $t \geq 0$. Define $T_f = T_f(\underline{\Delta}, \underline{\Delta}_{e_\xi}, \underline{\Delta}_{u_1}) > 0$ as

$$T_f := T_{A7}(\underline{\Delta}, \underline{\Delta}_{e_\xi}, \underline{\Delta}_{u_1}), \tag{C.19}$$

where T_{A7} is generated by Assumption 6.7. To apply Lemma 6.3, define $(\overline{\Delta}, \overline{\Delta}_{e_\xi}, \overline{\Delta}_{u_1})$ as $\overline{\Delta} := \underline{\Delta}$, $\overline{\Delta}_{e_\xi} := \underline{\Delta}_{e_\xi}$ and $\overline{\Delta}_{u_1} := \underline{\Delta}_{u_1}$. Hence, from the definition of $(\overline{\Delta}, \overline{\Delta}_{e_\xi}, \overline{\Delta}_{u_1})$, we use Lemma 6.3 to generate $\tilde{\varepsilon}^* > 0$. Define

$$k_{EP} := \frac{L_h^2}{p}, \tag{C.20}$$

where $L_h > 0$ is such that $|e_{y_i}(t_k + T_f) - e_{y_i}(t_k)|_\infty \leq \varepsilon L_h$ for any $t_k \geq 0$ (the existence of L_h is proven below). We now prove the statement of the lemma. Let us consider the error systems (6.34) and assume that $x(t)$ and $e_{x_i}(t)$ are frozen to $x(t_k)$ and $e_{x_i}(t_k)$, respectively, for $t_k \geq 0$. Hence, for $x(t) = x(t_k)$ and $e_{x_i}(t) = e_{x_i}(t_k)$, we denote the solution to the state error dynamics (6.34a) at time $t \geq t_k$ as $\overline{\varphi}(t, x(t_k), e_{x_i}(t_k))$, where we have omitted its dependence on ξ, u and $e_{\xi_i}(t_k)$. Similarly, we denote the error output (6.34b) at time $t \geq t_k$ as $\overline{\psi}(t, x(t_k), e_{x_i}(t_k))$. Since f_f and f_o are continuously differentiable, $e_{x_i}(t_k) \in \tilde{\mathbf{X}}$, $x(t_k) \in \mathbf{X}$, $\|u\|_\infty \leq \underline{\Delta}_u$ and since $|\xi(t)|_\infty \leq k_{A2}$ (by Assumption 6.2), it follows $\bar{f}_{e_i}$ is locally Lipschitz in φ, uniformly in x, ξ, e_{x_i}, and u by using similar arguments as in the proof of [Lemma 3.2, 70]. Hence, $\overline{\varphi}(t, x(t_k), e_{x_i}(t_k))$ is continuous in t, $x(t_k)$ and $e_{x_i}(t_k)$ by [Theorem 3.5, 70]. As a consequence, we deduce that $\overline{\psi}(t, x(t_k), e_{x_i}(t_k))$ is continuous in t, $x(t_k)$ and $e_{x_i}(t_k)$ by using the fact that h_e is continuously differentiable (Assumption 6.1). Now, consider the error systems (6.34) where $x(t)$ and $e_{x_i}(t)$ are time varying. Denote the solution to the state error dynamics (6.34a) at time $t \geq 0$ as $\varphi(t, x(t), e_{x_i}(t))$ and denote the error output (6.34b) at time $t \geq 0$ as $\psi(t, x(t), e_{x_i}(t))$ where we have omitted the dependency of these functions on ξ, u and $e_{\xi_i}(0)$. Then, as Lemma 6.3 hold, there is $k_{e_\xi} > 0$ generated from the choice of $(\overline{\Delta}, \overline{\Delta}_{e_\xi}, \overline{\Delta}_{u_1})$ so that $|e_{\xi_i}|_\infty \leq k_{e_\xi}$ for all $\varepsilon \in (0, \tilde{\varepsilon}^*)$. Hence, by following the same arguments as above, $\varphi(t, x(t), e_{x_i}(t))$ and $\psi(t, x(t), e_{x_i}(t))$ are continuous in t, $x(t)$ and $e_{x_i}(t)$. Since h is continuously differentiable, h, and subsequently, $\bar{h}_e$ in (6.34b) are Lipschitz on compact sets. Hence, it follows that there exists $L_h > 0$ such that, for $t \in [t_k, t_k + T_f]$,

$$|\psi(t, x(t), e_{x_i}(t)) - \overline{\psi}(t, x(t_k), e_{x_i}(t_k))|_\infty \leq \varepsilon L_h. \tag{C.21}$$

When $x(t)$ and $e_{x_i}(t)$ are frozen to $x(t_k)$ and $e_{x_i}(t_k)$, i.e. $x(t) = x(t_k)$ and $e_{x_i}(t) = e_{x_i}(t_k)$, the error output (6.34b) satisfies Assumption 6.7. Since $|e_{y_i}|_\infty \leq |e_{y_i}|$ where $|e_{y_i}|$ is the Euclidean norm of a vector $e_{y_i} \in \mathbb{R}^r$, it follows that (6.39) can be expressed in a scalar form as follows

$$\int_{t-T_f}^{t} (\upsilon^T e_{y_i}(\tau))^2 d\tau \geq \alpha_{A7}(|e_{x_i}(t_k)|_\infty), \tag{C.22}$$

for all $t \geq t_k + T_f$, with T_f defined as in (C.19), where $\upsilon \in \mathbb{R}^p$ is any constant vector with $|\upsilon| = 1$. As $e_{y_i}(t) = \overline{\psi}(t, x(t_k), e_{x_i}(t_k))$ when $x(t) = x(t_k)$ and $e_{x_i}(t) = e_{x_i}(t_k)$, it follows from (C.22) that

$$\int_{t-T_f}^{t} (\upsilon^T \overline{\psi}(t, x(t_k), e_{x_i}(t_k)))^2 d\tau = \int_{t-T_f}^{t} (\upsilon^T e_{y_i}(\tau))^2 d\tau \geq \alpha_{e_y}(|e_{x_i}(t_k)|_\infty). \tag{C.23}$$

By using (C.23), we have that

$$\int_{t-T_f}^{t} \psi(\tau)\psi^T(\tau) d\tau \geq \frac{1}{2}\int_{t-T_f}^{t} \overline{\psi}(\tau)\overline{\psi}^T(\tau) d\tau - \int_{t-T_f}^{t} [\psi(\tau) - \overline{\psi}(\tau)][\psi(\tau) - \overline{\psi}(\tau)]^T d\tau, \tag{C.24}$$

for all $t \geq T_f$, where we have omitted the dependency of ψ on $e_{x_i}(t)$ and $x(t)$ and the dependency of $\overline{\psi}$ on $e_{x_i}(t_k)$ and $x(t_k)$ to simplify notation. It follows from (C.21), (C.23) and (C.24) that

$$\int_{t-T_f}^{t} \psi(\tau, x(\tau), e_{x_i}(\tau))\psi^T(\tau, x(\tau), e_{x_i}(\tau)) d\tau \geq \frac{1}{2}\alpha_{A7}(|e_{x_i}(t_k)|_\infty) - L_h^2 \varepsilon^2. \tag{C.25}$$

Hence, it follows from the left-hand side of (C.25) that

$$\int_{t-T_f}^{t} \psi(\tau, x(\tau), e_{x_i}(\tau))\psi^T(\tau, x(\tau), e_{x_i}(\tau)) d\tau \geq \max\left\{\frac{1}{2}\alpha_{A7}(|e_{x_i}(0)|_\infty) - L_h^2 \varepsilon^2, 0\right\}. \tag{C.26}$$

Since $|e_{y_i}| \leq \sqrt{p}|e_{y_i}|_\infty$, we conclude from (C.26) that (6.42) holds with $\alpha_L(\cdot) \in \mathcal{K}_\infty$ and $k_{Pe} > 0$ defined as in (C.17) and (C.20), respectively. This completes the proof. ■

C.3 Proof of Lemma 6.5

Let $x(t), \hat{x}_i \in \mathbf{X}$ and let $\tilde{\Delta} > 0$, $\tilde{\Delta}_{e_\xi} > 0$, $\tilde{\Delta}_{u_1} > 0$ and $\nu > 0$ be given such that $|\xi(0)|_\infty \leq \tilde{\Delta}$, $|e_{\xi_i}(0)|_\infty \leq \tilde{\Delta}_{e_\xi}$ and $\|u\|_\infty \leq \tilde{\Delta}_{u_1}$. To apply Assumption 6.2, define (Δ, Δ_{u_1}) as $\Delta := \tilde{\Delta}$ and $\Delta_{u_1} := \tilde{\Delta}_{u_1}$. From the definition of $(\Delta, \Delta_{u_1}, \Delta_{u_2})$, we have that $k_{A2} > 0$ is generated by

Assumption 6.2. To apply Lemma 6.3, let define $(\overline{\Delta}, \overline{\Delta}_{e_\xi}, \overline{\Delta}_{u_1})$ as $\overline{\Delta} := \tilde{\Delta}$, $\overline{\Delta}_{e_\xi} := \tilde{\Delta}_{e_\xi}$ and $\overline{\Delta}_{u_1} := \tilde{\Delta}_{u_1}$. By using Lemma 6.3, we obtain $k_{L1} > 0$, $\lambda_{L1} > 0$ and $\tilde{\varepsilon}^* > 0$, and from the choice of $(\overline{\Delta}, \overline{\Delta}_{e_\xi}, \overline{\Delta}_{u_1})$, we generate $\gamma_L(\cdot) \in \mathcal{K}_\infty$. We now define $(\underline{\Delta}, \underline{\Delta}_{e_\xi}, \underline{\Delta}_{u_1})$ as $\underline{\Delta} := \tilde{\Delta}$, $\underline{\Delta}_{e_\xi} := \tilde{\Delta}_{e_\xi}$ and $\underline{\Delta}_{u_1} := \tilde{\Delta}_{u_1}$ so that Lemma 6.4 holds with these definitions. Since all sample points $\hat{x}_i \in \widehat{\mathbf{X}}$ belong to the set $\mathbf{X}$, there exists $k_{ex} > 0$ such that

$$|\hat{x}_i - x(t)|_\infty \leq k_{ex}, \tag{C.27}$$

for all $i \in \{1 \ldots, N\}$ and $t \geq 0$ where $N \in \mathbb{N}_{\geq 1}$. Then, it follows that $|e_{x_i}(t)|_\infty \leq k_{ex}$ for all $i \in \{1 \ldots, N\}$ and $t \geq 0$ so that Lemma 6.3 implies

$$\begin{aligned} |e_{\xi_i}(t)|_\infty &\leq k_{L1} \exp(-\lambda_{L_1} t)\, |e_{\xi_i}(0)|_\infty + \gamma_L(\|e_{x_i}\|_\infty) \\ &\leq k_{L1}\tilde{\Delta}_{e_\xi} + \gamma_L(k_{ex}), \end{aligned} \tag{C.28}$$

for all $\varepsilon \in (0, \tilde{\varepsilon}^*)$. Define a constant $k_{e_\xi} > 0$ as follows

$$k_{e_\xi} := k_{L1}\tilde{\Delta}_{e_\xi} + \gamma_L(k_{ex}). \tag{C.29}$$

Let $B_{k_{ex}} := \{e_{x_i} \in \mathbb{R}^n \mid |e_{x_i}|_\infty \leq k_{ex}\}$, $B_{k_{e_\xi}} := \{e_{z_i} \in \mathbb{R}^m \mid |e_{\xi_i}|_\infty \leq k_{e_\xi}\}$, and $B_{k_\xi} := \{\xi \in \mathbb{R}^m \mid |\xi|_\infty \leq k_{A2}\}$, where $|\xi|_\infty \leq k_{A2}$ follows from Assumption 6.2. Since the map h is continuously differentiable (Assumption 6.1), $x \in \mathbf{X}$, $e_{x_i} \in B_{k_{ex}}$, $\xi \in B_{k_\xi}$, and $e_{\xi_i} \in B_{k_{e_\xi}}$ (by (C.28)), h is locally Lipschitz. Moreover, we have that $\overline{h}_e(\xi(t), x(t), 0, 0, u(t)) = 0$ in view of the definition of $\overline{h}_e$ in (6.34). Therefore, there exist $\ell_{e_\xi} > 0$ and $\ell_{e_x} > 0$ such that

$$\begin{aligned} |\overline{h}_e(\xi(t), x(t), e_{\xi_i}(t), e_{x_i}(t), u(t))|_\infty &= |\overline{h}_e(\xi(t), x(t), e_{\xi_i}(t), e_{x_i}(t), u(t)) \\ &\quad - h_e(\xi(t), x(t), 0, 0, u(t))|_\infty \\ &= |h(e_{\xi_i}(t) + \xi(t), e_{x_i}(t) + x(t), u(t)) - h(\xi(t), x(t), u(t))|_\infty \\ &\leq |h(e_{\xi_i}(t) + \xi(t), e_{x_i}(t) + x(t), u(t)) - h(\xi(t), x(t), u(t))| \\ &\leq \ell_{e_x}|e_{x_i}(t)| + \ell_{e_\xi}|e_{\xi_i}(t)| \\ &\leq \sqrt{n}\ell_{e_x}|e_{x_i}(t)|_\infty + \sqrt{m}\ell_{e_\xi}|e_{\xi_i}(t)|_\infty. \end{aligned} \tag{C.30}$$

By using $\nu > 0$, define

$$\nu_\mu := \frac{1}{4}\nu, \tag{C.31}$$

$$\nu_{e_\xi} := \sqrt{\frac{\nu\lambda}{16m\ell_{e_\xi}^2}}, \tag{C.32}$$

where $\lambda > 0$ come from (6.9). Let $T_{e_\xi} > 0$ be sufficiently large such that

$$k_{L1} \exp\left(-\lambda_{L1} t\right) \tilde{\Delta}_{e_\xi} \leq \nu_{e_\xi}, \tag{C.33}$$

for all $t \geq T_{e_\xi}$ and for all $\varepsilon \in (0, \tilde{\varepsilon}^*)$. Let $T_\mu \geq T_{e_\xi}$ be sufficiently large such that

$$\frac{1}{\lambda} \exp(-\lambda(T_\mu - T_{e_\xi}))(\sqrt{m}\ell_{e_\xi} k_{e_\xi} + \sqrt{n}\ell_{e_x} k_{e_x})^2 \leq \nu_\mu. \tag{C.34}$$

Let the class-$\mathcal{K}_\infty$ function $\alpha_{L2}(\cdot)$, and the constants $k_{PE} > 0$ and $T_f > 0$ be generated by Lemma 6.4 such that the corresponding solutions to systems (6.1) and (6.34) satisfy

$$\int_{t-T_f}^{t} |e_{y_i}(s)|_\infty^2 ds \geq \max\left\{\alpha_{L2}(|e_{x_i}(0)|_\infty) - \varepsilon^2 k_{PE}, 0\right\}, \tag{C.35}$$

for all $t \geq T_f$. Define

$$k_{LM} := \exp(-\lambda T_f) k_{PE}^2, \tag{C.36}$$

$$\underline{\chi}(s) := \exp(-\lambda T_f)\alpha_{L2}(s), \tag{C.37}$$

$$\overline{\chi}(s) := \frac{4m\ell_{e_\xi}^2}{\lambda}\gamma_{L1}^2(2s) + \frac{4n\ell_{e_x}^2}{\lambda}s^2, \tag{C.38}$$

and $T(\tilde{\Delta}, \tilde{\Delta}_{e_\xi}, \tilde{\Delta}_{u_1}, \nu)$ as follows

$$T := \max\left\{T_\mu, T_{e_\xi}, T_f\right\}. \tag{C.39}$$

Note that $\underline{\chi}(\cdot), \overline{\chi}(\cdot) \in \mathcal{K}_\infty$ depend only on $\tilde{\Delta}$, $\tilde{\Delta}_{e_\xi}$, $\tilde{\Delta}_{u_1}$ and not on ν. Select $T_d > T$ such that $|e_{x_i}(t) - e_{x_i}(t_k)|_\infty \leq \varepsilon L_x T_d$ for all $t \in [t_k, t_{k+1})$ where $L_x > 0$ and $t_{k+1} - t_k = T_d$, $k \in \mathbb{N}$. Then, define

$$\overline{\varepsilon}^* = \min\{\overline{\varepsilon}_1^*, \overline{\varepsilon}_2^*, \overline{\varepsilon}_3^*\}, \tag{C.40}$$

where

$$\overline{\varepsilon}_1^* = \tilde{\varepsilon}^*, \tag{C.41}$$

$$\bar{\varepsilon}_2^* = \sqrt{\frac{\nu\lambda}{16\ell_{e_x}^2 L_x^2 T_d^2}}, \tag{C.42}$$

$$\bar{\varepsilon}_3^* = \frac{1}{2L_x T_d}\gamma_L^{-1}\left(\sqrt{\frac{\nu\lambda}{16mn\ell_{e_\xi}^2}}\right), \tag{C.43}$$

where $\tilde{\varepsilon}^*$ is generated by Lemma 6.3. Now, by using the definitions given above and following similar steps of proof of [Lemma 2, 25], we prove that (6.43) holds. Recall that the monitoring signals are periodically reset to zero after a finite time $T_d = t_{k+1} - t_k$. Let $t \geq t_k + T$ where T is defined as in (C.39). By definition of the monitoring signals in (6.9), we analyse

$$\mu_i(t_k, t) = \int_{t_k}^{t} \exp(-\lambda(t-s))|e_{y_i}(s)|_\infty^2 ds. \tag{C.44}$$

We first stablish the desired lower bound on the monitoring signals $\mu_i(\cdot,\cdot)$. To do so, consider

$$\begin{aligned}\mu_i(t_k, t) &= \int_{t_k}^{t-T_f} \exp(-\lambda(t-s))|e_{y_i}(s)|_\infty^2 ds + \int_{t-T_f}^{t} \exp(-\lambda(t-s))|e_{y_i}(s)|_\infty^2 ds \\ &\geq \int_{t-T_f}^{t} \exp(-\lambda(t-s))|e_{y_i}(s)|_\infty^2 ds.\end{aligned} \tag{C.45}$$

Since $s \mapsto \exp(-\lambda s)$ is strictly decreasing,

$$\mu_i(t_k, t) \geq \exp(-\lambda T_f) \int_{t-T_f}^{t} |e_{y_i}(s)|_\infty^2 ds. \tag{C.46}$$

As the monitoring signal are reset to zero at $t = t_k$, $k \in \mathbb{N}$, (C.35) leads to

$$\int_{t-T_f}^{t} |e_{y_i}(s)|_\infty^2 ds \geq \max\left\{\alpha_{L2}(|e_{x_i}(t_k)|_\infty) - \varepsilon^2 k_{PE}, 0\right\}, \tag{C.47}$$

for $t \geq t_k + T_f$. Hence, from (C.36), (C.37), (C.46) and (C.47), since $t \geq t_k + T_f$

$$\begin{aligned}\mu_i(t_k, t) &\geq \exp(-\lambda T_f) \max\left\{\alpha_{L2}(|e_{x_i}(t_k)|_\infty) - \varepsilon^2 k_{PE}, 0\right\} \\ &= \max\left\{\underline{\chi}(|e_{x_i}(t_k)|_\infty) - \varepsilon^2 k_{LM}, 0\right\}.\end{aligned} \tag{C.48}$$

We now obtain the desired upper bound for $\mu_i(\cdot,\cdot)$. Consider

$$\mu_i(t_k,t) = \int_{t_k}^{t_k+T_{e_\xi}} \exp(-\lambda(t-s))|e_{y_i}(s)|_\infty^2 ds + \int_{t_k+T_{e_\xi}}^{t} \exp(-\lambda(t-s))|e_{y_i}(s)|_\infty^2 ds. \quad (C.49)$$

It follows that

$$\mu_i(t_k,t) \leq \frac{1}{\lambda}\left[\exp(-\lambda(t-t_k-T_{e_\xi}))\right]\left(\sup_{s\in[t_k,t_k+T_{e_\xi}]}|e_{y_i}(s)|_\infty^2\right) + \int_{t_k+T_{e_\xi}}^{t} \exp(-\lambda(t-s))|e_{y_i}(s)|_\infty^2 ds. \quad (C.50)$$

Since $e_{y_i}(t) = \overline{h}_e(\xi(t), x(t), e_{\xi_i}(t), e_{x_i}(t), u(t))$, we use (C.30) and the fact that for any $\mathbf{a} \geq 0$, $\mathbf{b} \geq 0$, $(\mathbf{a}+\mathbf{b})^2 \leq 2\mathbf{a}^2 + 2\mathbf{b}^2$. Hence, we obtain

$$\begin{aligned}
\mu_i(t_k,t) &\leq \frac{1}{\lambda}\exp(-\lambda(t-t_k-T_{e_\xi}))\left(\sup_{s\in[t_k,t_k+T_{e_\xi}]}|e_{y_i}(s)|_\infty^2\right) \\
&\quad + \int_{t_k+T_{e_\xi}}^{t} \exp(-\lambda(t-s))\left(\sqrt{m}\ell_{e_\xi}|e_{\xi_i}(s)|_\infty + \sqrt{n}\ell_{e_x}|e_{x_i}(s)|_\infty\right)^2 ds \\
&\leq \frac{1}{\lambda}\exp(-\lambda(t-t_k-T_{e_\xi}))\left(\sup_{s\in[t_k,t_k+T_{e_\xi}]}|e_{y_i}(s)|_\infty^2\right) \\
&\quad + \int_{t_k+T_{e_\xi}}^{t} \exp(-\lambda(t-s))\left(2m\ell_{e_\xi}^2|e_{\xi_i}(s)|_\infty^2 + 2n\ell_{e_x}^2|e_{x_i}(s)|_\infty^2\right) ds. \quad (C.51)
\end{aligned}$$

Observe that for any $t \geq T_{e_\xi}$, it follows from Lemma 6.3 and (C.33) that

$$|e_{\xi_i}(t)|_\infty \leq \nu_{e_\xi} + \gamma_{L1}(\|e_{x_i}\|_\infty). \quad (C.52)$$

By using the property $(\mathbf{a}+\mathbf{b})^2 \leq 2\mathbf{a}^2 + 2\mathbf{b}^2$ for any $\mathbf{a} \geq 0$, $\mathbf{b} \geq 0$, it follows that

$$|e_{\xi_i}(t)|_\infty^2 \leq 2\nu_{e_\xi}^2 + 2\gamma_{L1}^2(\|e_{x_i}\|_\infty). \quad (C.53)$$

Moreover, as $e_{y_i}(t) = \overline{h}_e(\xi(t), x(t), e_{\xi_i}(t), e_{x_i}(t), u(t))$, by using (C.30) we obtain

$$\begin{aligned}
\sup_{s\in[t_k,t_k+T_{e_\xi}]}|e_{y_i}(s)|_\infty^2 &\leq \sup_{s\in[t_k,t_k+T_{e_\xi}]}(\sqrt{m}\ell_{e_\xi}|e_{\xi_i}(s)|_\infty + \sqrt{n}\ell_{e_x}|e_{x_i}(s)|_\infty)^2 \\
&= (\sqrt{m}\ell_{e_\xi}k_{e_\xi} + \sqrt{n}\ell_{e_x}k_{e_x})^2. \quad (C.54)
\end{aligned}$$

Since $\frac{1}{\lambda}\exp(-\lambda(t-t_k-T_{e_\xi})) \leq \frac{1}{\lambda}\exp(-\lambda(T_\mu - T_{e_\xi}))$ for all $t \geq t_k + T_\mu$, it follows from (C.34), (C.51), (C.53) and (C.54) that

$$\begin{aligned}\mu_i(t_k,t) &\leq \nu_\mu + \left(4m\ell_{e_\xi}^2\nu_{e_\xi}^2\right)\left(\int_{t_k+T_{e_\xi}}^t \exp(-\lambda(t-s))ds\right)\\ &+ \int_{t_k+T_{e_\xi}}^t \exp(-\lambda(t-s))\left(4m\ell_{e_\xi}^2\gamma_{L1}^2(\|e_{x_i}\|_\infty) + 2n\ell_{e_x}^2|e_{x_i}(s)|_\infty^2\right)ds. \end{aligned} \tag{C.55}$$

As $\int_{t_k+T_{e_\xi}}^t \exp(-\lambda(t-s))ds = \frac{1}{\lambda}(1-\exp(-\lambda(t-t_k-T_{e_\xi}))) \leq \frac{1}{\lambda}$, we conclude from (C.55) that

$$\begin{aligned}\mu_i(t_k,t) &\leq \nu_\mu + \frac{1}{\lambda}\left[\sup_{t\in[t_k+T_{e_\xi},t]}\left(4m\ell_{e_\xi}^2 n\gamma_{L1}^2(\|e_{x_i}\|_\infty) + 2\ell_{e_x}^2|e_{x_i}(t)|_\infty^2\right)\right]\\ &+ \frac{1}{\lambda}4m\ell_{e_\xi}^2\nu_{e_\xi}^2. \end{aligned} \tag{C.56}$$

The choice of $T_d > T$ implies $|e_{x_i}(t) - e_{x_i}(t_k)|_\infty \leq \varepsilon L_{e_x} T_d$ for all $t \in [t_k, t_{k+1})$. Hence, we obtain from (C.56) that

$$\begin{aligned}\mu_i(t_k,t) &\leq \nu_\mu + \frac{1}{\lambda}\left[\sup_{t\in[t_k+T_{e_\xi},t]}\left(4m\ell_{e_\xi}^2 n\gamma_{L1}^2\left(\sup_{t\in[t_k,t)}|e_{x_i}(t)|_\infty\right) + 2\ell_{e_x}^2|e_{x_i}(t)|_\infty^2\right)\right]\\ &+ \frac{1}{\lambda}4m\ell_{e_\xi}^2\nu_{e_\xi}^2. \end{aligned} \tag{C.57}$$

By adding and subtracting terms to (C.57) we obtain

$$\begin{aligned}\mu_i(t_k,t) &\leq \nu_\mu + \frac{1}{\lambda}\left[\sup_{t\in[t_k+T_{e_\xi},t]}\left(4m\ell_{e_\xi}^2 n\gamma_{L1}^2\left(\sup_{t\in[t_k,t)}|e_{x_i}(t) + e_{x_i}(t_k) - e_{x_i}(t_k)|_\infty\right)\right.\right.\\ &\left.\left. +2\ell_{e_x}^2|e_{x_i}(t) + e_{x_i}(t_k) - e_{x_i}(t_k)|_\infty^2\right)\right] + \frac{1}{\lambda}4m\ell_{e_\xi}^2\nu_{e_\xi}^2. \end{aligned} \tag{C.58}$$

Then, as $|e_{x_i}(t) - e_{x_i}(t_k)|_\infty \leq \varepsilon L_{e_x} T_d$ for all $t \in [t_k, t_{k+1})$, we conclude from (C.58) that

$$\begin{aligned}\mu_i(t_k,t) &\leq \nu_\mu + \frac{1}{\lambda}\left(4m\ell_{e_\xi}^2\nu_{e_\xi}^2 + 4m\ell_{e_\xi}^2 n\left[\gamma_{L1}^2\left(2\varepsilon L_{e_x}T_d\right) + \gamma_{L1}^2\left(2|e_{x_i}(t_k)|_\infty\right)\right]\right.\\ &\left. +4\ell_{e_x}^2\varepsilon^2 L_{e_x}^2 T_d^2 + 4\ell_{e_x}^2|e_{x_i}(t_k)|_\infty^2\right). \end{aligned} \tag{C.59}$$

By using $\nu_\mu, \nu_{e_\xi}, \overline{\chi}(\cdot) \in \mathcal{K}_\infty$ and $\overline{\varepsilon}^*$ in (C.31), (C.32), (C.38) and (C.40), we obtain

$$\mu_i(t_k,t) \leq \nu + \overline{\chi}(|e_{x_i}(t_k)|_\infty). \tag{C.60}$$

Therefore, (6.43) holds in view of (C.48) and (C.60). This completes the proof. ■

C.4 Proof of Theorem 6.2

Let $\Delta > 0, \Delta_{e_\xi} > 0, \Delta_u > 0, \tilde{\nu}_{e_x} > 0, \tilde{\nu}_{e_\xi} > 0$ be given an such that $|\xi(0)|_\infty \leq \Delta$, $|e_{\xi_i}(0)|_\infty \leq \Delta_{e_\xi}$, for $i \in \{1, \dots, N\}$, and $\|u\|_\infty \leq \Delta_u$. To apply Lemma 6.3, define $(\overline{\Delta}, \overline{\Delta}_{e_\xi}, \overline{\Delta}_{u_1})$ as $\overline{\Delta} := \Delta$, $\overline{\Delta}_{e_\xi} := \Delta_{e_\xi}$ and $\overline{\Delta}_{u_1} := \Delta_{u_1}$. By using Lemma 6.3, we obtain $\gamma_L(\cdot) \in \mathcal{K}_\infty$, $k_{L1} > 0$, $\lambda_{L1} > 0$ and $\tilde{\varepsilon}^* > 0$. Let $\eta > 0$ be sufficiently small such that

$$\eta \leq \gamma_L^{-1}\left(\tilde{\nu}_{e_\xi}\right), \tag{C.61}$$

where $\gamma_L(\cdot) \in \mathcal{K}_\infty$ are generated by Lemma 6.3. To apply Lemma 6.5, define $(\tilde{\Delta}, \tilde{\Delta}_{e_\xi}, \tilde{\Delta}_{u_1})$ as $\tilde{\Delta} := \Delta$, $\tilde{\Delta}_{e_\xi} := \Delta_{e_\xi}$ and $\tilde{\Delta}_{u_1} := \Delta_{u_1}$. From the definition of $(\tilde{\Delta}, \tilde{\Delta}_{e_\xi}, \tilde{\Delta}_{u_1})$, we generate $\underline{\chi}(\cdot), \overline{\chi}(\cdot) \in \mathcal{K}_\infty$, $k_{LM} > 0$, $T > 0$, $T_d > 0$ and $\overline{\varepsilon}^* > 0$ such that Lemma 6.5 holds with $\nu \in \left(0, \frac{1}{2}\underline{\chi}\left(\frac{1}{2}\min\{\eta, \tilde{\nu}_{e_x}\}\right)\right)$. It follows from compactness that there exists $\tilde{K}_{e_x} > 0$ such that

$$|\hat{x}_i - x(t)|_\infty \leq \tilde{K}_{e_x}. \tag{C.62}$$

Note that $\tilde{K}_{e_x}$ depends on the size of **X**. Hence, (C.62) implies that $|e_{x_i}(t)|_\infty \leq \tilde{K}_{e_x}$. We now define

$$\tilde{K}_{e_\xi} := k_{L1}\Delta_{e_\xi} + \gamma_L(\tilde{K}_{e_x}), \tag{C.63}$$

where $k_{L1} > 0$, and $\gamma_L(\cdot)$ come from Lemma 6.3 and $\tilde{K}_{e_x} > 0$ satisfies (C.62). Define

$$\hat{\varepsilon}^* := \min\{\hat{\varepsilon}_1^*, \hat{\varepsilon}_2^*, \hat{\varepsilon}_3^*, \hat{\varepsilon}_4^*\}, \tag{C.64}$$

where

$$\hat{\varepsilon}_1^* = \tilde{\varepsilon}^*, \tag{C.65}$$

$$\hat{\varepsilon}_2^* = \overline{\varepsilon}^*, \tag{C.66}$$

$$\hat{\varepsilon}_3^* = \sqrt{\frac{\frac{1}{2}\underline{\chi}\left(\frac{1}{2}\min\{\eta, \tilde{\nu}_{e_x}\}\right)}{k_{LM}}}, \tag{C.67}$$

$$\hat{\varepsilon}_4^* = \frac{\min\{\eta, \tilde{\nu}_{e_x}\}}{2L_x T_d}. \tag{C.68}$$

Then, define

$$d^* := \overline{\chi}^{-1}\left(\frac{1}{2}\underline{\chi}\left(\frac{1}{2}\min\{\eta, \tilde{\nu}_{e_x}\}\right) - \nu\right), \tag{C.69}$$

and note that $d^* > 0$ since $\nu \in \left(0, \frac{1}{2}\underline{\chi}\left(\frac{1}{2}\min\{\eta, \tilde{\nu}_{e_x}\}\right)\right)$. Hence, as $\max\left\{d\left(x, \widehat{\mathbf{X}}\right)\right\} \to 0$ as $N \to \infty$, it follows that there exists $N^* \in \mathbb{N}_{\geq 1}$ such that $\max\left\{d\left(x, \widehat{\mathbf{X}}\right)\right\} \leq d^*$ for all $N \geq N^*$. We now prove the statement of the theorem. It follows from (C.62) that

$$\max_{i\in\{1\ldots,N\}} |e_{x_i}(t)|_\infty \leq \tilde{K}_{e_x}. \tag{C.70}$$

Hence, we conclude from (C.70) that

$$|e_{x\sigma(t)}(t)|_\infty \leq \max_{i\in\{1\ldots,N\}} |e_{x_i}(t_k)|_\infty, \tag{C.71}$$

for all $t \geq 0$. Therefore, it follows from (C.70) and (C.71) that (6.44) holds. We now examine the state estimation error $e_{\xi,\sigma(t)}(t)$ for which we have

$$|e_{\xi\sigma(t)}(t)|_\infty \leq \max_{i\in\{1\ldots,N\}} |e_{\xi_i}(t)|_\infty. \tag{C.72}$$

By Lemma 6.3 and (C.62), we have that for all $i \in \{1\ldots,N\}$ the following holds

$$\begin{aligned} |e_{\xi_i}(t)|_\infty &\leq k_{L1}\exp\left(-\lambda_{L1}t\right)|e_{\xi_i}(0)|_\infty + \gamma_L(\|e_{x_i}\|_\infty) \\ &\leq k_{L1}\Delta_{e_\xi} + \gamma_L(\tilde{K}_{e_x}), \end{aligned} \tag{C.73}$$

for all $\varepsilon \in (0, \hat{\varepsilon}^*)$ where $\hat{\varepsilon}^* \leq \tilde{\varepsilon}^*$. Hence, we conclude from (C.63), (C.72) and (C.73) that the state estimation error satisfies (6.45). We now focus on the ultimate bounds in (6.46) and (6.47). Recall that we reset the monitoring signals to zero after a finite time $T_d = t_{k+1} - t_k$. Hence, by the selection criterion (6.12), we have that

$$\mu_{\sigma(t)} \leq \mu_i(t_k, t), \tag{C.74}$$

for all $t \in [t_k, t_{k+1})$ and for all $i \in \{1, \ldots, N\}$, where $\mu_{\sigma(t)}$ denotes the chosen monitoring signal for any $t \in [t_k, t_{k+1})$. Since (C.74) and Lemma 6.5 hold, it follows that

$$\max\left\{\underline{\chi}(|e_{x\sigma(t)}(t_k)|_\infty) - \varepsilon^2 k_{LM}, 0\right\} \leq \mu_{\sigma(t)} \leq \mu_{i^*}(t_k, t) \leq \overline{\chi}(|e_{x_{i^*}}(t_k)|_\infty) + \nu, \tag{C.75}$$

holds for all $t \in [t_k + T, t_{k+1})$ and $\varepsilon \in (0, \hat{\varepsilon}^*)$, where $\mu_{i^*}(t_k, t)$ denotes the monitoring signal with the smallest parameter estimation error, i.e. $i^* := \arg\min_{i \in \{1,\dots,N\}} |e_{x_{i^*}}(t)|_\infty$. We have that the smallest parameter estimation error satisfies $|e_{x_{i^*}}(t)|_\infty \leq \max\left\{ d\left(x, \widehat{\mathbf{X}}\right)\right\}$. Observe that $\sigma(t_k)$ is a piecewise constant function which is updated at $t = t_k$ so that $\sigma(t) = \sigma(t_k)$ for all $t \in [t_k, t_{k+1})$. It follows from (C.75) that

$$\begin{aligned} |e_{x\sigma(t_k)}(t_k)|_\infty &\leq \underline{\chi}^{-1}\left(\overline{\chi}(|e_{x_{i^*}}(t_k)|_\infty) + \nu + \varepsilon^2 k_{LM}\right) \\ &\leq \underline{\chi}^{-1}\left(\overline{\chi}\left(\max\left\{ d\left(x(t), \widehat{\mathbf{X}}\right)\right\}\right) + \nu + \varepsilon^2 k_{LM}\right), \end{aligned} \tag{C.76}$$

for all $\varepsilon \in (0, \hat{\varepsilon}^*)$. From the choice of $N^* \in \mathbb{N}_{\geq 1}$, it follows that for all $N \geq N^*$ the following holds

$$\begin{aligned} |e_{x\sigma(t_k)}(t_k)|_\infty &\leq \underline{\chi}^{-1}\left(\overline{\chi}\left(d^*\right) + \nu + \varepsilon^2 k_{LM}\right) \\ &= \underline{\chi}^{-1}\left(\frac{1}{2}\underline{\chi}\left(\frac{1}{2}\min\{\eta, \tilde{\nu}_{e_x}\}\right) + \varepsilon^2 k_{LM}\right). \end{aligned} \tag{C.77}$$

Hence, as $\hat{\varepsilon}^* \leq \hat{\varepsilon}_3^*$, we conclude that, for all $\varepsilon \in (0, \hat{\varepsilon}^*)$, the following holds

$$|e_{x\sigma(t_k)}(t_k)|_\infty \leq \frac{1}{2}\min\{\eta, \tilde{\nu}_{e_x}\}. \tag{C.78}$$

Observe that $e_{x\sigma(t_k)}(t) - e_{x\sigma(t_k)}(t_k) = \dot{e}_{x\sigma(t_k)} T_d$ for all $t \in [t_k, t_{k+1}]$. Since $\sigma(t) = \sigma(t_k)$ for all $t \in [t_k, t_{k+1}^-]$, we have that $e_{x\sigma(t_{k+1}^-)}(t_{k+1}^-) = e_{x\sigma(t_k)}(t_{k+1}^-)$. Hence, we obtain

$$e_{x\sigma(t_{k+1}^-)}(t_{k+1}^-) = \dot{e}_{x\sigma(t_k)} T_d + e_{x\sigma(t_k)}(t_k). \tag{C.79}$$

Then, by taking the ∞-norm on both sides of (C.79), we conclude that

$$|e_{x\sigma(t_{k+1}^-)}(t_{k+1}^-)|_\infty = |\dot{e}_{x\sigma(t_k)}|_\infty T_d + |e_{x\sigma(t_k)}(t_k)|_\infty. \tag{C.80}$$

Since $\dot{e}_{x_i} = \dot{x}$ for any $i \in \{1, \dots, N\}$, it follows that $|\dot{e}_{x\sigma(0)}|_\infty \leq \varepsilon L_x$ for a fixed $L_x > 0$. Then, (C.78) and (C.80) imply that

$$|e_{x\sigma(t_{k+1}^-)}(t_{k+1}^-)|_\infty \leq \varepsilon L_x T_d + \frac{1}{2}\min\{\eta, \tilde{\nu}_{e_x}\}. \tag{C.81}$$

It follows from (C.81) and the definition of $\hat{\varepsilon}_4^* > 0$ that

$$|e_{x\sigma(t_{k+1}^-)}(t_{k+1}^-)|_\infty \leq \min\{\eta, \tilde{\nu}_{e_x}\} \leq \tilde{\nu}_{e_x}, \tag{C.82}$$

for all $\varepsilon \in (0, \hat{\varepsilon}^*)$. Hence, it follows from (C.78) and (C.82) that (6.46) holds. We now examine the ∞-norm of the state estimation error $e_{\xi_{\sigma(t)}}(t)$. Since $\exp(-\alpha s)$ is a strictly decreasing function, it follows from (6.41) in Lemma 6.3 that

$$\limsup_{t\to\infty} |e_{\xi\sigma(t)}(t)|_\infty \leq \gamma_L\left(\min\{\eta, \tilde{\nu}_{e_x}\}\right) \leq \gamma_L(\eta). \tag{C.83}$$

Then, by (C.61), we conclude that

$$\limsup_{t\to\infty} |e_{\xi\sigma(t)}(t)|_\infty \leq \tilde{\nu}_{e_\xi}. \tag{C.84}$$

Hence, we conclude that (6.47) holds by virtue of (C.84). This completes the proof. ■

C.5 Proof of Lemma 6.6

Let conditions of Theorem 6.3 hold so that $|\xi(0)|_\infty \leq \hat{\Delta}$, $|e_{\xi_i}(0)|_\infty \leq \hat{\Delta}_{e_\xi}$ for $i \in \{1, \ldots, N\}$, and $\|u\|_\infty \leq \hat{\Delta}_{u_1}$. Then, by using Algorithm 6.3, we generate $\hat{\eta} > 0$, $\Delta_0 > 0$, $\Delta_1 > 0$, $\Delta_2 > 0$, $T^* > 0$, $N^* \in \mathbb{N}$, $N \geq N^*$, $T_d > 0$, and $\varepsilon^* > 0$. We now present the main elements we use in the proof. Recall from (6.50) that the selection criterion is defined as follows

$$\sigma(t) = \sigma(t_k) = \arg\min_{i\in\{1,\ldots,N\}} \mu_i(t_{k-1}, t_k), \tag{C.85}$$

for $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$. Indeed, as $T_d \geq T^*$, it follows from Algorithm 6.3 and Lemma 6.5 that

$$\max\{\underline{\chi}(|e_{x\sigma(t)}(t_k)|_\infty) - \varepsilon^2 k_{LM}, 0\} \leq \mu_{\sigma(t)} \leq \overline{\chi}(|e_{x\sigma(t)}(t_k)|_\infty) + \nu. \tag{C.86}$$

Let $e_{i^*}(t_k)$ be the parameter estimation error with the smallest norm so that

$$i^* = \arg\min_{i\in\{1,\ldots,N\}} |e_i(t_k)|_\infty.$$

Note that $|e_{\sigma(t_k)}(t_k)|_\infty$ is not necessarily equal to $|e_{i^*}(t_k)|_\infty$ for all t_k. Hence, it follows that, for $j \in \mathbb{N}$,

$$\mu_{\sigma(t_{k_{2j}})} \leq \mu_{i^*}(t_{k_{2j}-1}, t_{k_{2j}}). \tag{C.87}$$

Now, note that we have, for $k \in \mathbb{N}$, that the following holds

$$|x(t_{k+1}) - x(t_k)|_\infty < \pi(\Delta_2, N), \tag{C.88}$$

for all $\varepsilon \in (0, \varepsilon^*)$. This is verified in the following. Observe that (6.71) guarantees that

$$\varepsilon L_x T_d < \pi(\Delta_2, N), \tag{C.89}$$

for all $\varepsilon \in (0, \varepsilon^*)$. Since we have assumed $|\dot{x}(t)|_\infty \leq \varepsilon L_x$ for a fixed $L_x > 0$ and a sufficiently small $\varepsilon > 0$, it follows that the rate of change of the moving parameter is given by εL_x. Then, we have that

$$|x(t + T_d) - x(t)|_\infty \leq \varepsilon L_x T_d, \tag{C.90}$$

for all $t \geq 0$. Hence, as $T_d = t_{k+1} - t_k$, it follows from (6.71), (C.89) and (C.90) that (C.88) holds for all $\varepsilon \in (0, \varepsilon^*)$.

We now prove the result by dividing the rest of the proof in five steps. We first show that the first argument in the max term in (6.78) holds for all $j \in \mathbb{N}$. Secondly, by using the analysis from the first step, we prove that, for $j = 0$, $\Delta(t_k)$ becomes smaller than Δ_1 after a finite time . Then, in the third step, we show that (6.75) holds for all $j \in \mathbb{N}$. In the fourth step, we show that (6.76) and (6.77) hold too. Finally, in the fifth step we prove that the second argument in the max term in (6.78), and subsequently, (6.79) hold.

Step 1) For $j = 0$, we have that Algorithm 6.2 initialises $\Delta(t_0) = \Delta_0$ so that $\Delta_{in_0} = \Delta_0$. Then, the choice of N satisfying condition (6.66) guarantees $\mu_{\sigma(t_{k_0})} \leq \delta_0$. Hence, the switching law (6.51) and $\Delta_0 > \Delta_2$ imply that (6.75) holds at k_0. For $j \geq 1$, as the centre of the sampled set is moved to the best estimate, we have from (6.71) and (C.88) that $x(t_{k+1}^-) \in \overline{\mathbb{X}}(t_k)$ as long as $\Delta(t_k) \geq \Delta_2$; otherwise, a zoom-out may be triggered. Hence, as (6.74) and (C.88) imply that $\Delta_2 + 2\varepsilon L_x T_d \leq b\Delta_2$ for all $\varepsilon \in (0, \varepsilon^*)$, we have that Algorithm 6.3 ensures $x(t_{k_{2j}}^-) \in \overline{\mathbf{X}}(t_{k_{2j}-1})$. Moreover, from the definition of Δ_1 in Step 2 of Algorithm 6.3, we have that $\Delta(t_{k_{2j}-1}) = \Delta_{in_j} \in (\Delta_2, \Delta_1]$ for all $j \in \mathbb{N}_{\geq 1}$. Hence, we have that $x(t_{k_{2j}}^-) \in \overline{\mathbf{X}}(t_{k_{2j}-1})$, for $j \in \mathbb{N}$, so that (6.22) imply

$$|e_{x_{i^*}}(t_{k_{2j}}^-)|_\infty \leq \pi(\Delta(t_{k_{2j}-1}), N), \tag{C.91}$$

for all $j \in \mathbb{N}$. Then, for $j \geq 1$, we have that Lemma 6.5, the switching law (6.51), (6.67) and (C.87) imply

$$
\begin{aligned}
\mu_{\sigma(t_{k_{2j}})} &\leq \overline{\chi}(|e_{x_{i^*}}(t^-_{k_{2j}})|_\infty) + \nu \\
{}_{(C.91)} &\leq \overline{\chi}(\pi(\Delta(t_{k_{2j}-1}), N)) + \nu \\
&\leq \overline{\chi}(\pi(\Delta_1, N)) + \nu \;{}_{(6.67)} < \delta_1,
\end{aligned}
\tag{C.92}
$$

where the third inequality holds as $\Delta(t_{k_{2j}-1}) \in (\Delta_2, \Delta_1]$ Since $x(t^-_{k_{2j}}) \in \overline{\mathbf{X}}(t_{k_{2j}-1})$, for $j \in \mathbb{N}$, we have from (C.86) and (C.87) that

$$
\begin{aligned}
|e_{x\sigma(t^-_{k_{2j}})}(t^-_{k_{2j}})|_\infty &\leq \underline{\chi}^{-1}(\overline{\chi}(|e_{x_{i^*}}(t^-_{k_{2j}})|_\infty) + \nu + \varepsilon^2 k_{LM}) \\
{}_{(C.91)} &\leq \underline{\chi}^{-1}(\overline{\chi}(\pi(\Delta_{in}, N)) + \nu + \varepsilon^2 k_{LM}),
\end{aligned}
\tag{C.93}
$$

Note that (6.72) implies that $\varepsilon^2 k_{LM} \leq \nu$ for all $\varepsilon \in (0, \varepsilon^*)$, for small values of $k_{LM} > 0$. Hence, as $\Delta_{in_j} > \Delta_2$, for all $j \in \mathbb{N}$, we have from (6.65) and (C.93) that

$$
\begin{aligned}
|e_{x\sigma(t^-_{k_{2j}})}(t^-_{k_{2j}})|_\infty &\leq \underline{\chi}^{-1}(\overline{\chi}(\pi(\Delta_{in_j}, N)) + 2\nu) \\
{}_{(6.65)} &\leq a\Delta_{in_j} = \Delta(t_{k_{2j}}).
\end{aligned}
\tag{C.94}
$$

It follows from (C.94) that $x(t_{k_{2j}}) \in \mathbf{X}(x_c(t_{k_{2j}}), \Delta(t_{k_{2j}}))$ since $|x_c(t_{k_{2j}}) - x(t_{k_{2j}})|_\infty \leq \Delta(t_{k_{2j}})$ where $x_c(t_{k_{2j}}) = \hat{x}_{\sigma(t^-_{k_{2j}})}(t^-_{k_{2j}})$. Therefore, $x(t_{k_{2j}}) \in \overline{\mathbf{X}}(t_{k_{2j}})$. Moreover, if $\Delta(t_{k_{2j}}) \geq \Delta_2$, we conclude from $\pi(\Delta_2, N) \leq \Delta_2$, (C.89) and (C.90) that $x(t^-_{k_{2j+1}}) \in \overline{\mathbf{X}}(t_{k_{2j}})$ for any $\varepsilon \in (0, \varepsilon^*)$. Now, let $k \in [k_{2j}, k_{2j+1} - 1]$ and assume $\Delta(t_{k-2}) > \Delta_2$, $x(t^-_{k-1}) \in \overline{\mathbf{X}}(t_{k-2})$ and that

$$
|e_{x\sigma(t^-_{k-1})}(t^-_{k-1})|_\infty \leq a^{k-k_{2j}}\Delta_{in_j} = \Delta(t_{k-1}).
\tag{C.95}
$$

Then, if $\Delta(t_{k-1}) \geq \Delta_2$, we have that $\pi(\Delta_2, N) \leq \Delta_2$, (C.90) and (C.89) imply $x(t^-_k) \in \overline{\mathbf{X}}(t_{k-1})$. Hence, as in (C.91), it follows from (6.22) that $|e_{i^*}(t^-_k)|_\infty \leq \pi(\Delta(t_{k-1}), N)$ so that (C.86) and (C.87) lead to

$$
\begin{aligned}
|e_{x\sigma(t^-_k)}(t^-_k)| &\leq \underline{\chi}^{-1}(\overline{\chi}(|e_{i^*}(t^-_k)|_\infty) + \nu + \varepsilon^2 k_{LM}) \\
&\leq \underline{\chi}^{-1}(\overline{\chi}(\pi(\Delta(t_{k-1}), N)) + 2\nu),
\end{aligned}
\tag{C.96}
$$

for all $\varepsilon \in (0, \varepsilon^*)$, where the second inequality holds since $\varepsilon^2 K_{LM} \leq \nu$ for all $\varepsilon \in (0, \varepsilon^*)$, for small values of $k_{LM} > 0$. If $\Delta(t_{k-1}) \geq \Delta_2$, it follows from (6.65) and (C.96)

$$|e_{x\sigma(t_k^-)}(t_k^-)|_\infty \leq a\Delta(t_{k-1}) = \Delta(t_k). \tag{C.97}$$

Hence, we deduce from (C.94), (C.95) and (C.97) that $\Delta(t_k) = a^{k+1-k_{2j}}\Delta_{in_j}$, for $t_k \in [t_{k_{2j}}, t_{k_{2j+1}-1}]$, $j \in \mathbb{N}$, so that the first argument of $\max\{\cdot,\cdot\}$ on the right-hand side of (6.78) holds for all $j \in \mathbb{N}$. Moreover, (C.92) and (C.94) imply that $\Delta(t_{k_{2j}}) < \Delta_1$ for all $j \geq 1$. In the next step, we show that, for $j = 0$, there is a finite number of iterations before the parameter estimation error becomes bounded by Δ_1.

Step 2) We now consider $\Delta(t_k) = a^{k+1-k_{2j}}\Delta_{in_j}$ at $j = 0$ to characterise the transient time. Since $\Delta(t_k) = a^{k+1-k_0}\Delta_{in_0}$ is strictly decreasing in k, it follows that there exists $\tilde{k} > 0$ such that $a^{k+1-k_0}\Delta_{in_0} \geq \Delta_1$ for all $k \in [k_0, \tilde{k})$ and $a^{\tilde{k}+1-k_0}\Delta_{in_0} < \Delta_1$. It follows from the switching law (6.51) and (C.92) that $m(t_{\tilde{k}}) =$ 'zoom-in'.

Step 3) Since $\Delta(t_{\tilde{k}}) \leq \Delta_1$ for $j = 0$, $\Delta(t_{k_{2j}}) < \Delta_1$ for $j \geq 1$ and since $\Delta(t_k) = a^{k+1-k_{2j}}\Delta_{in_j}$ is strictly decreasing in k, there exist $\hat{k}_j > 0$, $j \in \mathbb{N}$, such that $a^{k+1-k_{2j}}\Delta_{in_j} \geq \Delta_2$ for all $k \in [k_{2j}, \hat{k}_j)$ and $a^{\hat{k}_j+1-k_{2j}}\Delta_{in_j} < \Delta_2$. Then, it follows from (6.71), (C.88) - (C.90) and (C.97) that $x(t_k) \in \overline{\mathbf{X}}(t_k)$ for all $k \in [k_{2j}, \hat{k}_j)$ and for all $\varepsilon \in (0, \varepsilon^*)$. Hence, as $x(t_k) \in \overline{\mathbf{X}}(t_k)$ for all $k \in [k_0, \hat{k}_0)$, we conclude from (6.66) and (6.67) that, at $j = 0$, $m(t_k) =$ 'zoom-in' for all $k \leq \hat{k}_0$ since $\Delta(t_k) \in [\Delta_2, \Delta_0]$ for all $k \leq \hat{k}_0$. Moreover, since $x(t_k) \in \overline{\mathbf{X}}(t_k)$ for all $k \in [k_{2j}, \hat{k}_j)$ for $j \geq 1$, we conclude from (6.67) that $m(t_k) =$ 'zoom-in' for all $k \in [k_{2j}, \hat{k}_j)$ as $\Delta(t_k) \in [\Delta_2, \Delta_1]$ for all $k \in [k_{2j}, \hat{k}_j)$. Therefore, we conclude that (6.75) holds for any $t_k \in [t_{k_{2j}}, t_{k_{2j+1}-1}]$.

Step 4) It follows from (6.43), (6.65), (C.89) and (C.90) that

$$|e_{x\sigma(t_{\hat{k}_0}^-)}(t_{\hat{k}_0}^-)|_\infty < \Delta_2. \tag{C.98}$$

If $m(t_k) =$ 'zoom-in' holds for any $t_k \geq t_{\hat{k}_0}$, then $\overline{\mathbf{X}}(t_k)$ keeps reducing its size and it may happen that, at some t_k, $x(t_k) \notin \overline{\mathbf{X}}(t_k)$ since $x(t)$ is moving. Then, it follows that (6.76) holds. When $x(t_k) \notin \overline{\mathbf{X}}(t_k)$, the lower bound in (6.43) may increase so that, at $t = t_{k_{2j+1}}$, $\mu_{\sigma(t_{k_{2j+1}})} > \delta_0$ and $m(t_{k_{2j+1}}) =$ 'zoom-out'. Hence, $m(t_{k_{2j+1}}) =$ 'zoom-out' implies $x(t_{k_{2j+1}}) \notin \overline{\mathbf{X}}(t_{k_{2j+1}})$ which only holds when $\Delta(t_{k_{2j+1}-1}) < \Delta_2$. Therefore, we conclude that (6.77) holds.

Step 5) We have from (6.51) in Algorithm 6.2 that a zoom-out is triggered if $\mu_{\sigma(t)} \geq \delta_0$. From the statement of Theorem 6.3, we have that $\delta_0 \in (0, \underline{\chi}((1-\theta)\Delta_2))$, for $\theta \in (0,1)$. Moreover, (6.73) in Algorithm 6.3 implies that $\underline{\chi}((1-\theta)\Delta_2) - \varepsilon^2 k_{LM} \geq \delta_0$ for all $\varepsilon \in (0, \varepsilon^*)$.

Hence, it follows from the lower bound in (C.86) that $\mu_{\sigma(t_k)} \geq \delta_0$ when $x(t_k) \notin \bar{\mathbf{X}}(t_k)$ and $|e_{x\sigma(t_k^-)}(t_k^-)|_\infty \geq (1-\theta)\Delta_2$. Then, a zoom out is triggered to prevent the parameter estimation error of increasing larger than Δ_2 so that we conclude

$$|e_{x\sigma(t_k^-)}(t_k^-)|_\infty < \Delta_2, \tag{C.99}$$

for all $t_k \in [t_{\tilde{k}_j}, t_{k_{2j+1}-1}]$, $j \in \mathbb{N}$. Therefore, we have from (C.97) - (C.99) that (6.78) holds. Furthermore, it follows from (C.99) that (6.79) holds for all $j \in \mathbb{N}$. This completes the proof. ■

C.6 Proof of Lemma 6.7

Let conditions of Theorem 6.3 hold so that $|\xi(0)|_\infty \leq \hat{\Delta}$, $|e_{\xi_i}(0)|_\infty \leq \hat{\Delta}_{e_\xi}$ for $i \in \{1, \ldots, N\}$, and $||u||_\infty \leq \hat{\Delta}_{u_1}$. Then, by using Algorithm 6.3, we generate $\hat{\eta} > 0$, $\Delta_0 > 0$, $\Delta_1 > 0$, $\Delta_2 > 0$, $T^* > 0$, $N^* \in \mathbb{N}$, $N \geq N^*$, $T_d > 0$, and $\varepsilon^* > 0$. The switching law (6.51) implies that $\mu_{\sigma(t_{k_{2j+1}-1})} \leq \delta_0$ and

$$\mu_{\sigma(t_{k_{2j+1}})} > \delta_0, \tag{C.100}$$

and subsequently, $m(t_{k_{2j+1}})$ = 'zoom-out'. Moreover, $\mu_{\sigma(t_k)} \geq \delta_1$ and $m(t_k)$ = 'zoom-out' for all $t_k \in [t_{k_{2j+1}}, t_{k_{2j+2}-1}]$, and

$$\mu_{\sigma(t_{k_{2j+2}})} < \delta_1, \tag{C.101}$$

which implies $m(t_{k_{2j+2}})$ = 'zoom-in'. Since $\Delta_{out} \in (0, \Delta_2)$ and $\Delta(t_{k_{2j+1}-1}) \leq \Delta_{out}$ for all $j \in \mathbb{N}$, we have that $\Delta(t_{k_{2j+1}-1}) < \Delta_2$. Lemma 6.6 implies that $m(t_{k_{2j+1}})$ = 'zoom-out' only happens when $\Delta(t_{k-1}) < \Delta_2$, and (6.79) implies that

$$|e_{x\sigma(t^-_{k_{2j+1}-1})}(t^-_{k_{2j+1}-1})|_\infty < \Delta_2, \tag{C.102}$$

Indeed, as $b > 1$ and $x_c(t_k) = x_c(t_{k_{2j+2}-1})$ for all $t_k \in [t_{k_{2j+1}}, t_{k_{2j+2}-1}]$, we have that

$$\mathbf{X}(x_c(t_{k_{2j+1}-1}), \Delta_2) \subset \mathbf{X}(x_c(t_k), b\Delta_2). \tag{C.103}$$

We know that $|\dot{x}(t)|_\infty \leq \varepsilon L_x$ implies that εL_x is the rate of change of the $x(t)$. Hence, to ensure $x(t_k) \in \mathbf{X}(x_c(t_k), b\Delta_2)$ for all $t_k \in [t_{k_{2j+1}}, t_{k_{2j+2}-1}]$, it follows from (C.102) and

(C.103) that ε has to be sufficiently small such that

$$\Delta_2 + 2\varepsilon T_d \leq b\Delta_2. \tag{C.104}$$

Note that (6.74) ensures that (C.104) holds for all $\varepsilon \in (0, \varepsilon^*)$ so that $x(t_k) \in \mathbf{X}(x_c(t_k), b\Delta_2)$ for all $t_k \in [t_{k_{2j+1}}, t_{k_{2j+2}-1}]$. A unit hypercube's longest diagonal in n dimensions is equal to $\sqrt{n}$. Since $x(t_k) \in \mathbf{X}(x_c(t_k), b\Delta_2)$, the largest possible distance from $x_{\sigma(t_k^-)}(t_k^-)$ to $x(t_k)$ is generated when the estimate and the real parameter are in opposite vertexes of the hypercube's longest diagonal. Therefore, we conclude that

$$|e_{x\sigma(t_k^-)}(t_k^-)|_\infty \leq 2b\sqrt{n}\Delta_2, \tag{C.105}$$

since $2b\Delta_2$ is the edge of the hypercube $\mathbf{X}(x_c(t_k), b\Delta_2)$. As $x(t_k) \in \mathbf{X}(x_c(t_k), b\Delta_2)$, it follows from (C.105) that $x(t_{k_{2j+1}}) \in \mathbf{X}(x_c(t_{k_{2j+1}}), 2b\sqrt{n}\Delta_2)$. Since $\Delta_1 = 2bc\sqrt{n}$ and $c = \Delta_2$, it follows $2b\sqrt{n}\Delta_2 = \Delta_1$. Hence, (6.43) and (6.67) imply

$$\mu_{\sigma(t_{k_{2j+1}+1})} < \delta_1, \tag{C.106}$$

which leads to $m(t_{k_{2j+2}+1}) =$ 'zoom-in'. Therefore, we conclude from (C.106) that $k_{2j+2} = k_{2j+1} + 1$ such that (6.80) holds. Moreover, $2b\sqrt{n}\Delta_2 = \Delta_1$ and (C.105) imply that (6.81) holds too. This completes the proof. ■

C.7 Proof of Theorem 6.3

Here, we use Algorithm 6.3 to construct the parameters of the statement of the theorem, and then we concatenate results from Lemmas 6.6 and 6.7. Let $\widehat{\Delta} > 0$, $\widehat{\Delta}_{e_\xi} > 0$, $\widehat{\Delta}_{u_1} > 0$, $\hat{\nu}_{e_x} > 0$, $\hat{\nu}_{e_\xi} > 0$, $a \in (0,1)$, $b > 1$, $c \in (0, \min\{\hat{\nu}_{e_x}, \gamma_L^{-1}(\hat{\nu}_{e_\xi})\}/2b\sqrt{n})$, $\delta_0 \in (0, \underline{\chi}((1-\theta)c))$, for $\theta \in (0,1)$, and $\delta_1 \in (0, \delta_0)$ be given, and let $|\xi(0)|_\infty \leq \widehat{\Delta}$, $|e_{\xi_i}(0)|_\infty \leq \widehat{\Delta}_{e_\xi}$ for $i \in \{1, \ldots, N\}$, and $\|u\|_\infty \leq \widehat{\Delta}_{u_1}$. Since Algorithm 6.2 guarantees that all sets $\overline{\mathbf{X}}(t_k) \subset \mathbf{X}$, for all $k \in \mathbb{N}$, we are able to invoke Lemmas 6.3 - 6.5. Moreover, it follows from compactness of $\mathbf{X} \subset \mathbb{R}^n$ that there exists $\widehat{K}_{e_x} > 0$ such that

$$|\hat{x}_i - x(t)|_\infty \leq \widehat{K}_{e_x}, \tag{C.107}$$

where $\widehat{K}_{e_x}$ depends on the size of **X**. Hence, (C.107) implies that $|e_{x_i}(t)|_\infty \leq \widehat{K}_{e_x}$. Define

$$\widehat{K}_{e_\xi} := k_{L1}\widehat{\Delta}_{e_\xi} + \gamma_L(\hat{K}_{e_x}), \tag{C.108}$$

where $k_{L1} > 0$ and $\gamma_L(\cdot)$ come from Lemma 6.3 and $\hat{K}_{e_x} > 0$ satisfies (C.107). We implement Algorithm 6.3 to generate $\hat{\eta} > 0$, $\Delta_0 > 0$, $\Delta_1 > 0$, $\Delta_2 > 0$, $T^* > 0$, $N^* \in \mathbb{N}$, $N \geq N^*$, $T_d > 0$, and $\varepsilon^* > 0$. We now prove that (6.60) and (6.61) hold. It follows from (C.107) that

$$\max_{i\in\{1\ldots,N\}} |e_{x_i}(t_k)|_\infty \leq \widehat{K}_{e_x}. \tag{C.109}$$

Since $\sigma(\cdot)$ in (6.50) is piecewise constant, we have that

$$|e_{x\sigma(t)}(t)|_\infty \leq \max_{i\in\{1\ldots,N\}} |e_{x_i}(t_k)|_\infty, \tag{C.110}$$

for all $t \geq 0$. Therefore, we conclude from (C.109) and (C.110) that (6.60) holds. We now examine the state estimation error $e_{\xi,\sigma(t)}(t)$. Similarly to the parameter estimation error, we have

$$|e_{\xi\sigma(t)}(t)|_\infty \leq \max_{i\in\{1\ldots,N\}} |e_{\xi_i}(t)|_\infty. \tag{C.111}$$

By Lemma 6.3 and (C.107), we have that for all $i \in \{1\ldots,N\}$ the following holds

$$\begin{aligned} |e_{\xi_i}(t)|_\infty &\leq k_{L1}\exp\left(-\lambda_{L1}t\right)|e_{\xi_i}(0)|_\infty + \gamma_L(\|e_{x_i}\|_\infty) \\ &\leq k_{L1}\widehat{\Delta}_{e_\xi} + \gamma_L(\hat{K}_{e_x}), \end{aligned} \tag{C.112}$$

for all $\varepsilon \in (0,\varepsilon^*)$ and $t \geq 0$, where $\varepsilon^* \leq \tilde{\varepsilon}^*$ with $\tilde{\varepsilon}^*$ generated by Lemma 6.3. Hence, we conclude from (C.108), (C.111) and (C.112) that (6.61) holds.

We now focus on the ultimate bounds in (6.62) and (6.63). Since we have implemented Algorithm 6.3, we are able to invoke Lemmas 6.6 and 6.7 which hold for all $\varepsilon \in (0,\varepsilon^*)$. We analyse the time interval $[t_{2j}, t_{2j+2})$ where both $m(t_k) =$ 'zoom-in' and $m(t_k) =$ 'zoom-out' have happened. We have from Lemma 6.6 that $\Delta(t_k) = a^k\Delta_0$ for all $t_k \in [t_{k_0}, t_{k_1-1}]$. Then, there exists $k_{inf} > 0$ such that $\Delta(t_k) \geq \hat{\eta}$ for all $k < k_{inf}$ and $\Delta(t_{k_{inf}}) < \hat{\eta}$ where k_{inf} is such that

$$k_{inf} > \left\lceil \frac{\ln \frac{\hat{\eta}}{\Delta_0}}{\ln a} \right\rceil, \tag{C.113}$$

where the right-hand side is positive since $\frac{\tilde{\eta}}{\Delta_0} < 1$ and $a < 1$. Moreover, it follows from Lemma 6.6 that, at $j = 0$, $\Delta(t_{k_1-1}) < \Delta_2$ and

$$|e_{x\sigma(t_{k_1-1})}(t_{k_1-1})|_\infty < \Delta_2. \tag{C.114}$$

Then, as $\Delta(t_{k_1-1}) < \Delta_2$, we are able to invoke Lemma 6.7 at $j = 0$ such that it follows from (6.81) that

$$|e_{x\sigma(t_{k_2-1})}(t_{k_2-1})|_\infty < \Delta_1. \tag{C.115}$$

We now proceed by induction. Let assume that

$$|e_{x\sigma(t)}(t)|_\infty < \Delta_1, \tag{C.116}$$

for all $t \in [t_{k_{2j}}, t_{k_{2j+2}})$ for $j \in \mathbb{N}_{\geq 1}$. Let $j = 1$. The construction of variables in Algorithm 6.3, Lemma 6.7 and (C.115) imply that $\Delta(t_{k_2-1}) < \Delta_1$ and $|e_{x\sigma(t^-_{k_2-1})}(t^-_{k_2-1})|_\infty \leq \Delta_1$. This follows from the fact that $\Delta_2 + 2\varepsilon T_d \leq b\Delta_2$ for all $\varepsilon \in (0, \varepsilon^*)$ and from $b\Delta_2 < \Delta_1$. Then, we can invoke Lemma 6.6 at $j = 1$ with $\Delta_{in} \in (\Delta_2, \Delta_1)$ so that (C.116) holds for all $t \in [t_{k_2}, t_{k_3-1}]$ and (6.79) implies

$$|e_{x\sigma(t^-_{k_3-1})}(t^-_{k_3-1})|_\infty < \Delta_2, \tag{C.117}$$

and $\Delta(t_{k_3-1}) < \Delta_2$ so that we are able to use results in Lemma 6.7 for $j = 1$. Therefore, it follows that (C.116) holds at t_{k_4-1}. Let consider the case $j + 1$. Since we assumed that (C.116) holds for all $t \in [t_{k_{2j}}, t_{k_{2j+2}})$ for $j \in \mathbb{N}_{\geq 1}$, we have $\Delta(t_{k_{2(j+1)}-1}) \leq \Delta_1$ and $|e_{x\sigma(t^-_{k_{2(j+1)}-1})}(t^-_{k_{2(j+1)}-1})|_\infty \leq \Delta_1$. Then, Lemma 6.6 implies (C.116) holds for all $t \in [t_{k_{2(j+1)}}, t_{k_{2(j+1)+1}-1}]$, and (6.79) leads to

$$|e_{x\sigma(t^-_{k_{2(j+1)+1}-1})}(t^-_{k_{2(j+1)+1}-1})|_\infty < \Delta_2, \tag{C.118}$$

and $\Delta(t_{k_{2(j+1)+1}-1}) < \Delta_2$. Then, we apply Lemma 6.7 at $j + 1$ so that (C.116) holds $t_{k_{2(j+1)+2}-1}$ by virtue of (6.81). Therefore, (C.116) holds for all $t \in [t_{k_{2j}}, t_{k_{2j+2}})$ for $j \in \mathbb{N}_{\geq 1}$. Since $\Delta_1 = 2bc\sqrt{n}$ and $c < \hat{n}/(2b\sqrt{n})$, it follows that $\Delta_1 < \hat{\eta}$ so that we conclude from (C.114), (C.115) and (C.116)

$$|e_{x\sigma(t_{k_2-1})}(t_{k_2-1})|_\infty < \hat{\eta}. \tag{C.119}$$

for all $\varepsilon \in (0, \varepsilon^*)$ and $t \geq t_{k_{inf}}$. Therefore, it follows from (6.64) and (C.119) that the following holds

$$\limsup_{t\to\infty} |e_{x\sigma(t)}(t)|_\infty \leq \hat{\eta} \leq \hat{\nu}_{e_x}. \tag{C.120}$$

We now examine the state estimation error $e_{\xi_{\sigma(t)}}(t)$. Since $\exp(-\lambda s)$ is a strictly decreasing function, it follows from (6.41) in Lemma 6.3, (C.111) and (C.120) that

$$\limsup_{t\to\infty} |e_{\xi\sigma(t)}(t)|_\infty \leq \gamma_L(\hat{\eta}). \tag{C.121}$$

Then, by (6.64), we conclude that

$$\limsup_{t\to\infty} |e_{\xi\sigma(t)}(t)|_\infty \leq \hat{\nu}_{e_\xi}. \tag{C.122}$$

Hence, we conclude that (6.62) and (6.63) hold for by virtue of (C.120) and (C.122). This completes the proof. ■

Bibliography

[1] V. Adetola and M. Guay, “Finite-time parameter estimation in adaptive control of nonlinear systems,” *IEEE Transactions on Automatic Control*, vol. 53, no. 3, pp. 807–811, 2008.

[2] A. Alessandri and A. Zemouche, “On the enhancement of high-gain observers for state estimation of nonlinear systems,” in *Proceedings of the 55th Conference on Decision and Control*, 2016, pp. 6278–6283.

[3] A. Alonso, I. G. Kevrekidis, J. R. Banga, and C. E. Frouzakis, “Optimal sensor location and reduced order observer design for distributed process systems,” *Computer and Chemical Engineering*, vol. 28, no. 1–2, pp. 27–35, 2004.

[4] V. Andrieu, L. Praly, and A. Astolfi, “High gain observers with updated gain and homogeneous correction terms,” *Automatica*, vol. 45, pp. 422–428, 2009.

[5] D. Angeli, E. D. Sontag, and Y. Wang, “A characterization of integral input-to-state stability,” *IEEE Transactions on Automatic Control*, vol. 45, no. 6, 2000.

[6] D. Angeli, E. D. Sontag, and Y. Wang, “Input-to-state stability with respect to inputs and their derivatives,” *International Journal of Robust and Nonlinear Control*, vol. 13, pp. 1035–1056, 2003.

[7] M. Arcak, “Circle-criterion observers and their feedback applications: An overview,” in *Current Trends in Nonlinear Systems and Control: In Honor of Petar Kokotović and Turi Nicosia*, L. Menini, L. Zaccarian, and C. Abdallah, Eds. Boston, MA: Birkhäuser Boston, 2006, pp. 3–14.

[8] M. Arcak and P. Kokotović, “Observer-based control of systems with slope-restricted nonlinearities,” *IEEE Transactions on Automatic Control*, vol. 4, no. 7, pp. 1146–1151, 2001.

[9] M. Arcak, “Unmodeled dynamics in robust nonlinear control,” Ph.D. dissertation, University of California, Santa Barbara, 2000.

[10] Z. Artstein, “Stability in the presence of singular perturbations,” *Nonlinear Analysis*, vol. 34, pp. 817–827, 1996.

[11] Z. Artstein and V. Gaitsgori, “Tracking fast trajectories along a slow dynamics, a singular perturbation approach,” *SIAM Journal on Control and Optimization*, vol. 35, pp. 1487–1507, 1997.

[12] A. Astolfi, R. Ortega, and A. Venkatraman, “Global observer design for mechanical systems with non-holonomic constraints,” in *Proceedings of American Control Conference*, Baltimore USA, 2010.

[13] A. Astolfi and L. Praly, “Global complete observability and output-to-state stability imply the existence of a globally convergent observer,” in *Proceedings of IEEE Conference on Decision and Control*, Hawaii USA, 2003.

[14] D. Astolfi and L. Marconi, “A high-gain nonlinear observer with limited gain power,” *IEEE Transactions on Automatic Control*, vol. 60, no. 11, pp. 3059–3064, 2015.

[15] G. Battistelli, J. P. Hespanha, and P. Tesi, “Supervisory control of switched nonlinear systems,” *International Journal of Adaptive Control and Signal Processing*, vol. 26, no. 8, pp. 723–738, 2012.

[16] G. Besançon, *Nonlinear Observers and Applications.* Berlin Heidelberg: Springer-Verlag, 2007.

[17] D. Bestle and M. Zeitz, “Canonical form observer design for nonlinear time variable systems,” *International Journal of Control*, vol. 38, pp. 419–431, 1983.

[18] B. S. Bhangu, P. Bentley, D. A. Stone, and C. M. Bingham, “Nonlinear observers for predicting state-of-charge and state-of-health of lead-acid batteries for hybrid-electric vehicles,” *IEEE Transactions on Vehicular Technology*, vol. 54, no. 3, pp. 783–794, 2005.

[19] E. Biyik and M. Arcak, “Area aggregation and time scale modeling for sparse nonlinear networks,” in *Proceedings of IEEE Conference on Decision and Control*, 2006, pp. 4046–4051.

[20] J. J. Bongiorns and D. C. Youla, “On observers in multivariable control systems,” *International Journal of Control*, vol. 8, pp. 221–243, 1968.

[21] F. Bornemann, *Homogenization in Time of Singularly Perturbed Mechanical Systems.* New York: Springer-Verlag, 1998.

[22] A. D. Bruno, “Singular perturbation in hamiltonian mechanics,” *Hamiltonian Mechanics: Integrability and Chaotic Behaviour*, pp. 43–49, 1994.

[23] D. Carnevale, D. Karagiannis, and A. Astolfi, “Reduced-order observer design for systems with non-monotonic nonlinearities,” in *Proceedings of IEEE Conference on Decision and Control*, 2006, pp. 5260–5274.

[24] R. Castro-Linares, J. Alvarez-Gallegos, and V. Vazquez-Lopez, “Sliding mode control and state estimation for class of nonlinear singularly perturbed systems,” *Dynamics and Control*, vol. 11, no. 1, pp. 25–46, 2001.

[25] M. S. Chong, D. Nešić, R. Postoyan, and L. Kuhlmann, “Parameter and state estimation of nonlinear systems using a multi-observer under the supervisory framework,” *IEEE Transactions on Automatic Control*, vol. 60, no. 9, pp. 2336–2349, 2015.

[26] M. S. Chong, R. Postoyan, D. Nešić, L. Kuhlmann, and A. Varsavsky, “A circle criterion observer for estimating the unmeasured membrane potential of neuronal populations,” in *Proceedings of 2011 Australian Control Conference*, 2011.

[27] M. S. Chong, R. Postoyan, D. Nešić, L. Kuhlmann, and A. Varsavsky, “A robust circle criterion observer with application to neural mass models,” *Automatica*, vol. 48, no. 11, pp. 2986–2989, 2012.

[28] P. D. Christofides, “Output feedback control of nonlinear two time-scale systems,” in *Proceedings of the American Control Conference*, 1997, pp. 1729—1733.

[29] P. D. Christofides, “Output feedback control of nonlinear two time-scale processes,” *Industrial and Engineering Chemestry*, vol. 37, pp. 1893–1909, 1998.

[30] P. D. Christofides, “Robust output feedback control of nonlinear singularly perturbed systems,” *Automatica*, vol. 36, no. 1, pp. 45–52, 2000.

[31] P. D. Christofides and P. Daoutidis, “Feedback control of two-time-scale nonlinear systems,” *International Journal of Control*, vol. 63, no. 5, pp. 965–994, 1996.

[32] P. D. Christofides and P. Daoutidis, "Distributed output feedback control of two time-scale hyperbolic PDE systems," *Applied Mathematics and Computer Science*, vol. 8, no. 4, pp. 713–732, 1998.

[33] P. D. Christofides and A. R. Teel, "Singular perturbations and input-to-state stability," *IEEE Transactions on Automatic Control*, vol. 41, no. 11, pp. 1645–1650, 1996.

[34] P. D. Christofides, A. R. Teel, and P. Daoutidis, "Robust semi-global output tracking for nonlinear singularly perturbed systems," in *Proceedings of the 34th Conference on Decision and Control*, 1995, pp. 2251—2256.

[35] M. Corless and L. Glielmo, "On the exponential stability of singularly perturbed systems," *SIAM Journal on Control and Optimization*, vol. 30, no. 6, pp. 1338–1360, 1991.

[36] N. Daroogheh, N. Meskin, and K. Khorasani, "Robust hybrid EKF approach for state estimation in multi-scale nonlinear singularly perturbed systems," in *Proceedings of IEEE Conference on Decision and Control*, 2014, pp. 1047–1054.

[37] J. H. Davis, *Fundations of Deterministic and Stochastic Control.* Birkhäuser, Boston, MA: Systems and control: Fundations and Applications, 2002.

[38] M. A. Demetriou and N. Kazantzis, "Natural observer design for singularly perturbed vector second-order systems," *Transactions of the ASME*, vol. 127, pp. 648–655, 2005.

[39] M. Dimitriev and G. Kurina, "Singular perturbations in control problems," *Automation and Remote Control*, vol. 67, no. 1, pp. 1–43, 2006.

[40] P. Ducommun, A. Kadouri, U. von Stockar, and I. W. Marison, "On-line determination of animal cell concentration in two industrial high-density culture processes by dielectric spectroscopy," *Biotechnology and Bioengineering*, vol. 77, no. 3, pp. 316–323, 2002.

[41] T. Edison and A. Michael, "Observer theory for continuous-time linear systems," *Information and Control*, vol. 22, no. 5, pp. 405–434, 1973.

[42] G. Ellis, *Observers in Control Systems.* Academic Press, 2002.

[43] X. Fan and M. Arcak, “Nonlinear observer design for systems with multivariable monotone nonlinearities,” in *Proceedings of IEEE Conference on Decision and Control*, Las Vegas USA, 2002.

[44] Z. Gajic and M. Lim, “A new filtering method for linear singularly perturbed systems,” *IEEE Transactions on Automatic Control*, vol. 39, no. 9, pp. 1952–1955, 1994.

[45] Z. Gajic and M. Lim, *Optimal Control of Singularly Perturbed Linear Systems and Applications: High-accuracy Techniques*. New York, USA: Marcel Dekker, Inc., 2001.

[46] I. T. Georgiou, “On the global geometric structure of the dynamics of the elastic pendulum,” *Nonlinear Dynamics*, vol. 18, no. 1, pp. 51–68, 1999.

[47] G. Gilboa, R. Chen, and N. Brenner, “History-dependent multiple-time-scale dynamics in single-neuron model,” *The Journal of Neuroscience*, vol. 25, no. 28, pp. 6479–6489, 2005.

[48] L. Grne, E. D. Sontag, and F. R. Wirth, “Asymptotic stability equals exponential stability, and ISS equals finite energy gain—if you twist your eyes,” *Systems and Control Letters*, vol. 38, no. 2, pp. 127–134, 1999.

[49] A. H. Haddad, “Linear filtering of singularly perturbed systems,” *IEEE Transactions on Automatic Control*, vol. 21, pp. 515–519, 1976.

[50] W. M. Haddad and V. Chellaboina, *Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach*. New Jersey: Princeton University Press, 2008.

[51] J. P. Hespanha, D. Liberzon, and A. S. Morse, “Hysteresis-based switching algorithms for supervisory control of uncertain systems,” *Automatica*, vol. 39, pp. 263–272, 2003.

[52] M. H. Holmes, *Introduction to Perturbation Methods*, 2nd ed. New York: Springer, 1998.

[53] F. H. Hsiao, J. D. Hwang, and S. T. Pan, “Stabilization of discrete singularly perturbed systems under composite observer-based control,” *Journal of Dynamic Systems, Measurement, and Control*, vol. 123, pp. 132–139, 2001.

[54] C. Hu, B. D. Youn, and J. Chung, "A multi-scale framework with Extended Kalman Filter for lithium-ion battery SOC and capacity estimation," *Applied Energy*, vol. 92, pp. 694–704, 2012.

[55] P. A. Ioannou, *Robust Adaptive Control.* Prentice Hall, 1996.

[56] A. Isidori, S. S. Sastry, P. V. Kokotovic, and C. I. Byrnes, "Singularly perturbed zero dynamics of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 37, no. 10, 1992.

[57] B. H. Jansen and V. G. Rit, "Electroencephalogram and visual evoked potential generation in a mathematical model of coupled cortical columns," *Biologival Cybernetics*, vol. 73, pp. 357–366, 1995.

[58] S. H. Javid, "Observing the slow states of a singularly perturbed system," *IEEE Transactions on Automatic Control*, vol. 25, pp. 277–280, 1980.

[59] S. H. Javid, "Stabilization of time-varying singularly perturbed systems by observer-based slow-state feedback," *IEEE Transactions on Automatic Control*, vol. 27, no. 3, pp. 702–704, 1982.

[60] Z. P. Jiang, A. R. Teel, and L. Praly, "Small-gain theorem for ISS systems and applications," *Mathematics of Control, Signals, and Systems*, vol. 7, pp. 95–120, 1994.

[61] A. Johansson and A. Medvedev, "An observer for systems with nonlinear output map," *Automatica*, vol. 39, pp. 909–918, 2003.

[62] S. J. Julier and J. K. Uhlmann, "New extension of the Kalman filter to nonlinear systems," in *Proceedings of SPIE 3068, Signal Processing, Sensor Fusion and Target Recognition V1*, 1997, pp. 6278–6283.

[63] R. E. Kalman, "A new approach to linear filtering and prediction problems," *Tnasactions of the ASME Journal of Basic Engineering*, vol. 82, no. 1, pp. 35–45, 1960.

[64] R. E. Kalman and R. S. Bucy, "New results in linear filtering and preduction theory," *Tnasactions of the ASME Journal of Basic Engineering*, vol. 83, no. 1, pp. 155–160, 1961.

[65] H. Kando, T. Aoyama, and T. Iwazumi, "Observer design of singularly perturbed discrete systems via time-scale decomposition approach," *International Journal of Systems Science*, vol. 21, no. 5, pp. 815–833, 1990.

[66] D. Karagiannis, D. Carnevale, and A. Astolfi, "Invariant manifold based reduced-order observer design for nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 53, no. 11, pp. 2602–2614, 2008.

[67] D. Karagiannis, M. Sassano, and A. Astolfi, "Dynamic scaling and observer design with application to adaptive control," *Automatica*, vol. 45, pp. 2883–2889, 2009.

[68] N. Kazantzis, N. Huynh, and R. A. Wright, "Nonlinear observer design for the slow states of singularly perturbed systems," *Computer and Chemical Engineering*, vol. 29, pp. 797–806, 2005.

[69] N. Kazantzis and C. Kravaris, "Nonlinear observer design using Lyapunov's auxiliary theorem," *Signal and Control Letters*, vol. 34, no. 5, pp. 241–247, 1998.

[70] H. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, New Jersey: Prentice Hall, 2001.

[71] H. Khalil, *High-Gain Observers in Nonlinear Feedback Control*, 1st ed. Society of Industrial and Applied Mathematics, 2017.

[72] J. Kim and B. H. Cho, "State-of-charge estimation and state-of-health prediction of a li-ion degraded battery based on an EKF combined with a per-unit system," *IEEE Transactions on Vehicular Technology*, vol. 60, no. 9, pp. 4249–4260, 2011.

[73] J. W. Kimball and P. T. Krein, "Singular perturbation theory for DC-DC converters and application to PFC converters," *IEEE Transactions on Power Electronics*, vol. 23, pp. 2970–2981, 2008.

[74] P. Kokotović, J. Chow, and H. Khalil, "Singularly perturbed systems," *Wiley Encyclopedia of Electrical and Electronics Engineering*, 1999.

[75] P. Kokotović, H. Khalil, and J. O'Reilly, *Singular Perturbation Methods in Control: Analysis and Design.* New York: Society for Industrial and Applied Mathematics, 1999.

[76] A. Krener and A. Isidori, "Linearization by output injection and nonlinear observers," *Systems and Control Letters*, vol. 3, pp. 47–52, 1983.

[77] A. Kumar, P. D. Christofides, and P. Daoutidisia, "Singular perturbation modelling of nonlinear processes with nonexplicit time-scale multiplicity," *Chemical Engineering Science*, vol. 53, no. 8, pp. 1491–1504, 1998.

[78] J. Lee, J. Hong, R. O. K. Nam, L. Praly, and A. Astolfi, “Sensorless control of surface-mount permanent-magnet synchronous motors based on nonlinear observer,” *IEEE Transactions on Power Electronics*, vol. 25, no. 2, pp. 290–297, 2010.

[79] J. H. Li and T. H. Li, “On the composite and reduced observer-based control of discrete two-time-scale systems,” *Journal of the Franklin Institute*, vol. 332, pp. 47–66, 1995.

[80] D. Liberzon and D. Nešić, “Input-to-state stabilization of linear systems with quantized state measurements,” *IEEE Transactions on Automatic Control*, vol. 52, no. 5, pp. 767–781, 2007.

[81] K. J. Lin, “Composite observer-based feedback design for singularly perturbed systems via LMI approach,” in *Proceedings of SICE Annual Conference 2010*, 2010, pp. 3056–3061.

[82] H. P. Liu, F. C. Sun, and K. Z. He, “Survey of singularly perturbed control systems: Theory and applications,” *Control Theory and Applications*, vol. 20, no. 1, pp. 1–7, 2003.

[83] L. Ljung, *System Identification.* Wiley Online Library, 1999.

[84] W. Lohmiller and J. J. Slotine, “On contraction analysis of nonlinear systems,” *Automatica*, vol. 34, pp. 683–696, 1998.

[85] D. G. Luenberger, “Observing the state of a linear system,” *IEEE Transactions on Military Electronics*, vol. 8, pp. 74–80, 1964.

[86] D. G. Luenberger, “Observers for multivariable systems,” *IEEE Transactions on Automatic Control*, vol. 11, pp. 90–97, 1966.

[87] J. Luo, K. R. Pattipati, L. Qiao, and S. Chigusa, “Model-based prognostic techniques applied to a suspension system,” *IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans*, vol. 38, no. 5, pp. 1156–1168, 2008.

[88] M. S. Mahmoud and H. K. Khalil, “Rubostness of high-gain observer-based nonlinear controllers to unmodeled actuators and sensors,” *Automatica*, vol. 38, no. 2, pp. 361–369, 2002.

[89] W. L. Miranker, *Numerical Methods for Stiff Equations and Singular Perturbation Problems.* Dordrecht: D. Reifel Pub. Co., 1981.

[90] A. S. Morse, “Supervisory control of families of linear set-point controllers - Part 1: Exact matching,” *IEEE Transactions on Automatic Control*, vol. 41, no. 10, pp. 1413–1431, 1996.

[91] A. S. Morse, “Supervisory control of families of linear set-point controllers - Part 2: Robustness,” *IEEE Transactions on Automatic Control*, vol. 42, no. 11, pp. 1500–1515, 1997.

[92] A. S. Morse, “Estimator-based supervisory control: A review,” *Systems, Control and Information*, vol. 42, no. 6, pp. 319–326, 1998.

[93] D. S. Naidu and A. J. Calise, “Singular perturbation and time scales in guidance and control of aerospace systems: A survey,” *Journal of Guidance, Control, and Dynamics*, vol. 24, no. 6, pp. 1057–1078, 2001.

[94] D. Nešić, “Extremum seeking control: Convergence analysis,” in *Proceedings of the 2009 European Control Conference (ECC)*, 2009, pp. 1702–1715.

[95] D. Nešić and P. Dower, “A note on input-to-state stability and averaging of systems with inputs,” *IEEE Transactions on Automatic Control*, vol. 46, no. 11, pp. 1760–1765, 2001.

[96] D. Nešić and A. R. Teel, “Input-to-state stability for nonlinear time-varying systems via averaging,” *Mathematics of Control, Signals, and Systems*, vol. 14, pp. 257–280, 2001.

[97] L. Nie and Z. Teng, “Singular perturbation method for global stability of ratio-dependent predator-prey models with stage structure for the prey,” *Electronic Journal of Differential Equations*, vol. 2013, no. 86, pp. 1–9, 2013.

[98] H. Nijmeijer and T. I. Fossen, *New Directions in Nonlinear Observer Design*. London: Springer-Verlag, 1999.

[99] M. Okun, N. A. Steinmetz, M. D. A. Lak, and K. D. Harris, “Distinct structure of cortical population activity on fast and infraslow timescales,” *Cerebral Cortex*, vol. 29, no. 5, pp. 2196–2210, 2019.

[100] H. Oloomi and M. E. Sawan, “The observer-based controller design of discrete-time singularly perturbed systems,” *IEEE Transactions on Automatic Control*, vol. 32, no. 3, pp. 246–248, 1987.

[101] J. O'Reilly, "Full-order observers for a class of singularly perturbed linear time-varying systems," *International Journal of Control*, vol. 30, no. 5, pp. 745–756, 1979.

[102] J. O'Reilly, "Dynamical feedback control for a class of singularly perturbed linear systems using a full-order observer," *International Journal of Control*, vol. 31, pp. 1–10, 1980.

[103] R. Ortega, L. Praly, A. Astolfi, J. Lee, and K. Nam, "Estimation of rotor position and speed of permanent magnet synchronous motors with guaranteed stability," *IEEE Transactions on Control Systems Technology*, vol. 19, no. 3, pp. 601–614, 2011.

[104] G. Phanomchoeng, R. Rajamani, and D. Piyabongkarn, "Nonlinear observer for bounded Jacobian systems, with applications to automotive slip angle estimation," *IEEE Transactions on Automatic Control*, vol. 56, no. 5, pp. 1163–1170, 2011.

[105] L. Praly, "Convergence theory for observers: Necessary, and sufficient conditions," 2016, summary of lectures, European Embedded Control Institute, International Graduate School on Control 2016.

[106] A. Saberi and H. K. Khalil, "Quadratic-type Lyapunov functions for singularly perturbed systems," *IEEE Transactions on Automatic Control*, vol. 29, no. 6, pp. 542–550, 1984.

[107] D. Saha and J. Valasek, "Observer-based sequential control of a nonlinear two-time-scale system with multiple slow and fast states," in *Proceedings of 10th IFAC Symposium on Nonlinear Control and Systems*, vol. 49, no. 18, 2016, pp. 696–701.

[108] D. Saha and J. Valasek, "Observer-based sequential control of a nonlinear two-time-scale spring-mass-damper system," in *AIAA Guidance, Navigation, and Control Conference*, 2016.

[109] R. G. Sanfelice and L. Praly, "Nonlinear observer design with appropriate riemannian metric," in *Proceedings of IEEE Conference on Decision and Control*, 2009, pp. 6514–6519.

[110] R. G. Sanfelice and L. Praly, "Convergence of nonlinear observers on $\mathbb{R}^n$ with a riemannian metric (Part 1)," *IEEE Transactions on Automatic Control*, vol. 57, pp. 1709–1722, 2012.

[111] A. V. Sebald and A. H. Haddad, "State estimation for singularly perturbed systems with uncertain perturbation parameter," *IEEE Transactions on Automatic Control*, vol. 23, pp. 464–469, 1978.

[112] R. Sharma, D. Nešić, and C. Manzie, "Model reduction of a turbocharged (TC) spark ignition (SI) engines," *IEEE Transactions on Control and Systems Technology*, vol. 19, no. 2, pp. 297–310, 2011.

[113] K. R. Shouse and D. G. Taylor, "Discrete-time observers for singularly perturbed continuous-time systems," *IEEE Transactions on Automatic Control*, vol. 40, no. 2, pp. 224–235, 1995.

[114] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Transactions on Automatic Control*, vol. 34, no. 4, pp. 435–442, 1989.

[115] E. D. Sontag, "New characterizations of input-to-state stability," *IEEE Transactions on Automatic Control*, vol. 41, no. 9, pp. 1283–1294, 1996.

[116] E. D. Sontag and Y. Wang, "On characterizations of input-to-state stability with respect to compact sets," *Systems and Control Letters*, 1995.

[117] H. W. Sorenson, *Kalman Filtering: Theory and Applications*. IEEE Press, 1985.

[118] A. R. Teel, L. Moreau, and D. Nešić, "A unified framework for input-to-state stability in systems with two time scales," *IEEE Transactions on Automatic Control*, vol. 48, no. 9, pp. 1526–1544, 2003.

[119] F. Thau, "Observing the state of non-linear dynamic systems," *International Journal of Control*, vol. 17, pp. 471–479, 1973.

[120] D. Upadhyay and M. Nieuwstadt, "Modeling of a urea SCR catalyst with automotive applications," in *ASME International Mechanical Engineering Congress & Exposition*, 2002.

[121] F. Verhulst, *Methods and Applications of Singular Perturbations: Boundary Layers and Multiple Time-scale Dynamics*. The Netherlands: Springer, 2005.

[122] L. Vu and D. Liberzon, "Supervisory control of uncertain linear time-varying systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 1, pp. 27–42, 2011.

[123] Y. Wang and W. Liu, "Robust observer-based feedback control for Lipschitz singularly perturbed systems," *Mathematical Problems in Engineering*, 2015.

[124] C. Yang, L. Zhang, and L. Zhou, "Observer design for singularly perturbed systems with multirate sampled and delayed measurements," *Journal of Dynamic Systems, Measurement, and Control*, vol. 138, no. 5, pp. 51 007–1–51 007–9, 2016.

[125] H. Yoo and Z. Gajic, "New designs of reduced-order observer-based controllers controllers for singularly perturbed linear systems," *Mathematical Problems Engineering*, pp. 1–14, 2017.

[126] H. Yoo and Z. Gajic, "New designs of linear observers and observer-based controllers for singularly perturbed linear systems," *IEEE Transactions on Automatic Control*, vol. 63, no. 11, pp. 3904–3911, 2018.

[127] H. Yoo, "New methods for design of full and reduced order observers and observer-based controllers for systems with slow and fast modes," Ph.D. dissertation, The State University of New Jersey, New Brunswick, 2017.

[128] A. Zemouche, R. Rajamani, G. Phanomchoeng, and B. Boulkroune, "Circle criterion-based $\mathcal{H}_\infty$ observer design for Lipschitz and monotonic nonlinear systems - enhanced LMI conditions and constructive discussions," *Automatica*, vol. 85, pp. 412–425, 2017.

[129] Y. Zhang, D. S. Naidu, C. Cai, and Y. Zou, "Singular perturbation and time scales in control theories and applications: An overview 2002-2012," *Automation and Remote Control*, vol. 67, no. 1, pp. 1–43, 2014.

[130] C. Zou, "Modelling, state estimation & optimal charging control for a lithium-ion battery," Ph.D. dissertation, The University of Melbounre, 2016.

[131] Y. Zou, X. Hu, H. Ma, and S. E. Li, "Combined state of charge and state of health estimation over lithium-ion battery cell cycle lifespan for electric vehicles," *Journal of Power Sources*, vol. 273, no. 1, pp. 793–803, 2015.